2. $\dfrac{\partial w}{\partial y}\dfrac{\partial}{\partial x}(\Delta w)-\dfrac{\partial w}{\partial x}\dfrac{\partial}{\partial y}(\Delta w)=\nu\Delta\Delta w,\qquad \Delta w=\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}.$

There is a ***two-dimensional stationary equation of motion of a viscous incompressible fluid*** (it is obtained from the Navier–Stokes equations by eliminating pressure and introducing the stream function w, see Remark).

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the functions

$$
\begin{aligned}
w_1&=-w(y,\,x),\\
w_2&=w(C_1x+C_2,\,C_1y+C_3)+C_4,\\
w_3&=w(x\cos\alpha+y\sin\alpha,\,-x\sin\alpha+y\cos\alpha),
\end{aligned}
$$

where $C_1,\,\dots,\,C_4$ and α are arbitrary constants, are also solutions of the equation.

2°. Any solution of the Poisson equation $\Delta w=C$ is also a solution of the original equation (these are "inviscid" solutions).

3°. Solutions in the form of a one-variable function or the sum of functions with different arguments:

$$
\begin{aligned}
w(y)&=C_1y^3+C_2y^2+C_3y+C_4,\\
w(x,y)&=C_1x^2+C_2x+C_3y^2+C_4y+C_5,\\
w(x,y)&=C_1\exp(-\lambda y)+C_2y^2+C_3y+C_4+\nu\lambda x,\\
w(x,y)&=C_1\exp(\lambda x)-\nu\lambda x+C_2\exp(\lambda y)+\nu\lambda y+C_3,\\
w(x,y)&=C_1\exp(\lambda x)+\nu\lambda x+C_2\exp(-\lambda y)+\nu\lambda y+C_3,
\end{aligned}
$$

where $C_1,\,\dots,\,C_5$ and λ are arbitrary constants.

4°. Generalized separable solutions:

$$
\begin{aligned}
w(x,y)&=A(kx+\lambda y)^3+B(kx+\lambda y)^2+C(kx+\lambda y)+D,\\
w(x,y)&=Ae^{-\lambda(y+kx)}+B(y+kx)^2+C(y+kx)+\nu\lambda(k^2+1)x+D,\\
w(x,y)&=6\nu x(y+\lambda)^{-1}+A(y+\lambda)^3+B(y+\lambda)^{-1}+C(y+\lambda)^{-2}+D,\\
w(x,y)&=(Ax+B)e^{-\lambda y}+\nu\lambda x+C,\\
w(x,y)&=\bigl[A\sinh(\beta x)+B\cosh(\beta x)\bigr]e^{-\lambda y}+\frac{\nu}{\lambda}(\beta^2+\lambda^2)x+C,\\
w(x,y)&=\bigl[A\sin(\beta x)+B\cos(\beta x)\bigr]e^{-\lambda y}+\frac{\nu}{\lambda}(\lambda^2-\beta^2)x+C,\\
w(x,y)&=Ae^{\lambda y+\beta x}+Be^{\gamma x}+\nu\gamma y+\frac{\nu}{\lambda}\gamma(\beta-\gamma)x+C,\qquad \gamma=\pm\sqrt{\lambda^2+\beta^2},
\end{aligned}
$$

where A, B, C, D, k, β, and λ are arbitrary constants.

5°. Generalized separable solution linear in x:

$$w(x,y)=F(y)x+G(y),\qquad (1)$$

where the functions $F=F(y)$ and $G=G(y)$ are determined by the autonomous system of fourth-order ordinary differential equations

$$F'_yF''_{yy}-FF'''_{yyy}=\nu F''''_{yyyy},\qquad (2)$$
$$G'_yF''_{yy}-FG'''_{yyy}=\nu G''''_{yyyy}.\qquad (3)$$

Equation (2) has the following particular solutions:

$$F = ay + b,$$
$$F = 6\nu(y+a)^{-1},$$
$$F = ae^{-\lambda y} + \lambda\nu,$$

where a, b, and λ are arbitrary constants.

Let $F = F(y)$ is a solution of equation (2) ($F \not\equiv \text{const}$). Then, the corresponding general solution of equation (3) can be written out in the form

$$G = \int U\,dy + C_4, \qquad U = C_1U_1 + C_2U_2 + C_3\left(U_2\int \frac{U_1}{\Phi}\,dy - U_1\int \frac{U_2}{\Phi}\,dy\right),$$

where C_1, C_2, C_3, and C_4 are arbitrary constants, and

$$U_1 = \begin{cases} F''_{yy} & \text{if } F''_{yy} \not\equiv 0, \\ F & \text{if } F''_{yy} \equiv 0, \end{cases} \qquad U_2 = U_1\int \frac{\Phi\,dy}{U_1^2}, \qquad \Phi = \exp\left(-\frac{1}{\nu}\int F\,dy\right).$$

6°. There is an exact solution of the form (generalizes the solution of Item 5°):

$$w(x,y) = F(z)x + G(z), \qquad z = y + kx, \qquad k \text{ is any}.$$

7°. Self-similar solution:

$$w = \int F(z)\,dz + C_1, \qquad z = \arctan\left(\frac{x}{y}\right),$$

where the function F is determined by the first-order autonomous ordinary differential equation $3\nu(F'_z)^2 - 2F^3 + 12\nu F^2 + C_2F + C_3 = 0$ (C_1, C_2, and C_3 are arbitrary constants).

8°. There is an exact solution of the form (generalizes the solution of Item 7°):

$$w = C_1 \ln|x| + \int V(z)\,dz + C_2, \qquad z = \arctan\left(\frac{x}{y}\right).$$

Remark. The two-dimensional steady-state equations of a viscous incompressible fluid (the Navier–Stokes equations)

$$u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\Delta u_1,$$
$$u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_2}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\Delta u_2,$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0$$

are reduced to the equation in question by the introduction of a stream function w such that $u_1 = \frac{\partial w}{\partial y}$ and $u_2 = -\frac{\partial w}{\partial x}$ followed by the elimination of the pressure p (with cross differentiation) from the first two equations.

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