Napier's ideal construction of the logarithms*

Denis Roegel

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1 Introduction

Today John Napier (1550–1617) is most renowned as the inventor of logarithms.[1] He had conceived the general principles of logarithms in 1594 or before and he spent the next twenty years in developing their theory [109, p. 63], [33, pp. 103–104]. His description of logarithms, *Mirifici Logarithmorum Canonis Descriptio*, was published in Latin in Edinburgh in 1614 [132, 162] and was considered "one of the very greatest scientific discoveries that the world has seen" [84]. Several mathematicians had anticipated properties of the correspondence between an arithmetic and a geometric progression, but only Napier and Jost Bürgi (1552–1632) constructed tables for the purpose of simplifying the calculations. Bürgi's work was however only published in incomplete form in 1620, six years after Napier published the *Descriptio* [26].[2]

Napier's work was quickly translated in English by the mathematician and cartographer Edward Wright[3] (1561–1615) [146, 180] and published posthumously in 1616 [133, 163]. A second edition appeared in 1618. Wright was a friend of Henry Briggs (1561–1630) and this in turn may have led Briggs

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[1] Among his many activities and interests, Napier also devoted a lot of time to a commentary of Saint John's Revelation, which was published in 1593. One author went so far as writing that Napier "invented logarithms in order to speed up his calculations of the Number of the Beast." [41]

[2] It is possible that Napier knew of some of Bürgi's work on the computation of sines, through Ursus' *Fundamentum astronomicum* (1588) [150].

[3] In 1599, prior to Napier, logarithms had actually been used implicitely by Wright, but without Wright realizing that he had done so, and without using them to simplify calculations [33]. Therefore, Wright can not (and did not) lay claim on a prior discovery. See Wedemeyer's article [202] for additional information.

to visit Napier in 1615 and 1616 and further develop the decimal logarithms. Briggs' first table of logarithms, *Logarithmorum chilias prima*, appeared in 1617 [20] and contained the logarithms in base 10 of the first 1000 (*chilias*) integers to 14 places. It was followed by his *Arithmetica logarithmica* in 1624 [21] and his *Trigonometria Britannica* in 1633 [22].

The details of Napier's construction of the logarithms were published posthumously in 1619 in his *Mirifici Logarithmorum Canonis Constructio* [134, 135, 137]. Both the *Descriptio* and the *Constructio* have had several editions.[4]

It should be stressed that Napier's logarithms were not the logarithms known today as Neperian (or natural, or hyperbolic) logarithms, but a related construction made at a time when the integral and differential calculus did not yet exist. Napier's construction was wholly geometrical and based on the theory of proportions.

We have recomputed the tables published in 1614 and 1616 according to Napier's rules,[5] using the GNU `mpfr` library [55]. The computations have been done as if Napier had made no error.

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[4] Latin editions of the *Descriptio* were published in 1614 [132], 1619, 1620 [135] (*google books*, id: `QZ4_AAAAcAAJ`), 1658, 1807 [111], and 1857. English editions were published in 1616, 1618, and 1857 (by Filipowski) [136]. Latin editions of the *Constructio* were published in 1619 [134], 1620 [135] (*google books*, id: `VukHAQAAIAAJ`, `UJ4_AAAAcAAJ`), and 1658. Some of the copies of the edition published in 1620 in Lyon carry the year 1619. The 1658 edition is a remainder of the 1619–1620 editions, with title page and preliminary matter reset [109]. An English edition of the *Constructio* was published in 1889 [137]. In addition, Archibald cites a 1899 edition of the two Latin texts [4, p. 187], and he probably refers to Gravelaar's description of Napier's works [72], but as a matter of fact, the latter only contains excerpts of the *Descriptio* and *Constructio*. More recently, W. F. Hawkins also gave English translations of both works [79] in his thesis. Then, Jean Peyroux published a French translation of the *Descriptio* in 1993 [97]. Finally, Ian Bruce made new translations which are available on the web at `http://www.17centurymaths.com`. For extensive details on the various editions of Napier's works known by 1889, see Macdonald [137, pp. 101–169].

[5] Throughout this article, we consider an idealization of Napier's calculations, assuming that some values were computed exactly. In fact, Napier's values were rounded, and the present results would be slightly different in a presentation of Napier's *actual* construction. A very detailed analysis of these problems was given by Fischer [50]. We might provide a table based on Napier's rounding methods in the future.

2 Napier's construction

Napier was interested in simplifying[6] computations and he introduced a new notion of numbers which he initially called "artificial numbers." This name is used in his *Mirifici logarithmorum canonis constructio*, published in 1619, but written long before his *descriptio* of 1614. In the latter work, Napier introduced the word *logarithm*, from the Greek roots *logos* (ratio) and *arithmos* (number). A number of authors have interpreted this name as meaning the "number of ratios,"[7] but G. A. Gibson and W. R. Thomas believe that the word logarithm merely signifies "ratio-number" or "number connected with ratio" [189, p. 195]. Moreover, Thomas believes that Napier found the expression "number connected with ratio" in Archimedes' *Sand Reckoner*, for it is known that Napier was a good Greek scholar and a number of princeps editions of Archimedes' book were available in England by the early 17th century [189, 87].

In any case, Napier meant to provide an arithmetic measure of a (geometric) ratio, or at least to define a continuous correspondence between two progressions. Napier took as origin the value 10^7 and defined its logarithm to be 0. Any smaller value x was given a logarithm corresponding to the ratio between 10^7 and x.

2.1 First ideas

It is likely that Napier first had the idea of a correspondence between a geometric series and an arithmetic one. Such correspondences have been exhibited before, in particular by Archimedes, Nicolas Chuquet in 1484, or not long before Napier by Stifel [185]. According to Cantor, Napier's words for negative numbers are the same as those of Stifel, which leads him to think that he was acquainted with Stifel's work [35, p. 703]. Bürgi was also

[6]It seems that Napier had heard from the method of *prosthaphaeresis* for the simplification of trigonometric computations through John Craig who had obtained it from Paul Wittich in Frankfurt at the end of the 1570s [63, pp. 11–12]. The same Wittich also influenced Tycho Brahe and Jost Bürgi, who came close to discovering logarithms [167]. This connection with John Craig has been misinterpreted as meaning that Longomontanus invented logarithms which would then have been copied by Napier [33, p. 99–101]. On the unfairness of several authors towards Napier, see Cajori [33, pp. 107–108]. Perhaps the extreme case of unfairness is that of Jacomy-Régnier who claims that Napier had his calculating machine made by Bürgi himself and that in return the shy Bürgi told him of his invention of logarithms [89, p. 53].

[7]For example, Carslaw is misled by the apparent meaning of "logarithm," although he observes that Napier's construction does not absolutely agree with this meaning [37, p. 82].

indirectly influenced by Stifel and, in fact, the most extensive correspondence between an arithmetic and a geometric progression was probably that of Bürgi [26]. But, as we will see, Napier could impossibly go the whole way of an explicit correspondence, and he needed another definition.

2.2 The definition of the logarithm

A precise definition of the logarithm was given by Napier as follows.[8] He considered two lines (figure 1), with two points moving from left to right at different speeds. On the first line, the point α is moving arithmetically from left to right, that is, with a constant speed equal to 10^7. The figure shows the positions of the point at instant T, $2T$, etc., where T is some unit of time. The first interval is traversed in $10^7 T$. In modern terms, $x_\alpha(t) = 10^7 t$.

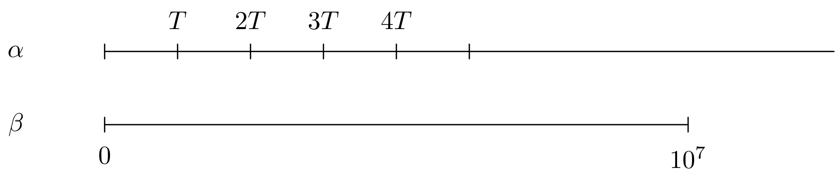


Figure 1: The progressions α and β.

On the second line (which is of length 10^7), the point β is moving geometrically, in the sense that the distances left to traverse at the same instants T, $2T$, etc. form a geometrical sequence. Let us analyze Napier's definition in modern terms. If d_i is the remaining distance at instant t_i, we have $\frac{d_{i+1}}{d_i} = c$, where c is some constant. If $x_\beta(t)$ is the distance traversed since the beginning, we have

$$\frac{10^7 - x_\beta((n+1)T)}{10^7 - x_\beta(nT)} = c$$

If we set $y_\beta(t) = 10^7 - x_\beta(t)$, we have $y_\beta((n+1)T) = c y_\beta(nT)$ and therefore $y_\beta(nT) = c^n y_\beta(0)$. It follows that $y_\beta(t) = c^{t/T} y_\beta(0) = Aa^t$, for t equal to multiples of T.

Initially ($t = 0$), point β is at the left of the line, and therefore $A = 10^7$. It follows that

$$x_\beta(t) = 10^7(1 - a^t) = 10^7(1 - e^{t \ln a})$$

Again, this is normally only defined for $t = kT$.

[8] We have borrowed some of the notations from Michael Lexa's introduction [104].

Napier set the initial speed to 10^7 and this determines the motion entirely. Indeed, from $x'_\beta(t) = -10^7 a^t \ln a$, we easily obtain $a = e^{-1}$. The initial speed therefore determines a. Moreover $x_\beta(t)$ does not depend on T, which means that Napier's construction defines a correspondence where the unit T plays no role. The motion defined by Napier therefore corresponds to the equation

$$x_\beta(t) = 10^7(1 - e^{-t})$$

but it should be remembered that Napier nowhere uses these notations, and that there were no exponentials or derivatives in his construction. One particular consequence (which was Napier's assumption) of this definition is that the speed of β is equal to the distance left to traverse:

$$x'_\beta(t) = 10^7 a^t = 10^7 - x_\beta(t) = y_\beta(t).$$

Another consequence is that point β travels for equal time increments between any two numbers that are equally proportioned, and this will be used later when computing the logarithms of the second table constructed by Napier.

Now, Napier's definition of the logarithm, which we denote λ_n, is the following: if at some time, β is at position x and α at position y, then $\lambda_n(10^7 - x) = y$. For instance, at the beginning, $x = y = 0$ and $\lambda_n(10^7) = 0$.

Using a kinematic approach and the theory of proportions,[9] Napier was thus able to define a correspondence between two continua, those of lines α and β.

The modern expression for Napier's logarithms can easily be obtained. To $y_\beta(t)$ Napier associates $x_\alpha(t)$, or in other words, to $x = 10^7 e^{-t}$, Napier associates $y = \lambda_n(x) = 10^7 t$, hence

$$\lambda_n(x) = 10^7(\ln(10^7) - \ln x). \tag{1}$$

In particular, $\frac{\lambda_n(x)}{10^7} - \ln(10^7)$ is the modern logarithm in base $\frac{1}{e}$, but, contrary to what has been written by various authors, $\lambda_n(x)$ is not the logarithm in base $\frac{1}{e}$. However, if instead of the factor 10^7, we take 1, then $\lambda_n(x)$ is indeed the modern logarithm of base $\frac{1}{e}$.[10]

[9] Many works have been concerned with proportions before Napier, and we can mention in particular the Portuguese Alvarus Thomas who in 1509 used a geometric progression to divide a line [188, 33].

[10] For an overview of various bases which have been ascribed to Napier's logarithms, see Matzka [112, 113], Cantor [35, p. 736], Tropfke [191, p. 150], and Mautz [114, pp. 6–7]. Many other authors addressed this question, and the reader should consult the references cited at the end of this article.

But we can also write $\frac{\lambda_n(x)}{10^7} = -\ln\left(\frac{x}{10^7}\right)$ and we can view 10^7 as a scaling factor, both for $\lambda_n(x)$ and for x. In other words, if we scale both the sines and the logarithms in Napier's tables down by 10^7, we obtain a correspondence which is essentially the logarithm of base $\frac{1}{e}$.

Although the logarithms are precisely defined by Napier's procedure, their computation is not obvious. This, *per se*, is already particularly interesting, as Napier managed to define a numerical concept, and to separate this concept from its actual computation.

Next, Napier defines geometric divisions which he will use to approximate the values of the logarithms.

2.3 Geometric divisions

Napier then constructed three geometric sequences. These sequences all start with 10^7 and then use different multiplicative factors. These tables are given in sections 10.1, 10.2, and 10.3.[11]

The first sequence a_i uses the ratio $r_1 = 0.9999999$ such that $\frac{a_{i+1}}{a_i} = r_1$ and $a_0 = 10^7$. Napier computes 100 terms of the sequence using exactly 7 decimal places, until $a_{100} = 9999900.0004950$ [137, p. 13]. The values of a_i are given in section 10.1. Napier's last value is correct.

The second sequence b_i also starts with 10^7, but uses the ratio $r_2 = 0.99999$. We have $\frac{b_{i+1}}{b_i} = r_2$. Napier computes 50 terms until he reaches $b_{50} = 9995001.224804$ using exactly 6 decimal places. Napier's value was actually $b'_{50} = 9995001.222927$ [137, p. 14] and he must have made a computation error, as the error is much larger than the one due to rounding.[12] This error, together with the confidence Napier displays for his calculations, certainly indicates that he alone did the computations.[13]

The first and second sequences are related in that $b_1 \approx a_{100}$, which follows from $0.9999999^{100} \approx 0.99999$. However, as we will see, Napier does not identify a_{100} and b_1 nor $\lambda_n(a_{100})$ and $\lambda_n(b_1)$.

[11]Some of our notations for the sequences appear to be identical, or nearly identical, to those of Carslaw [37, pp. 79–81], but this is a mere coincidence. For some reason, Carslaw did not name the values of the third table, except those of the first line and first column.

[12]Even without the details of Napier's calculations, and even without Napier's manuscript, it is probably possible to pinpoint the error more precisely, assuming that only one error was made, but we have not made any attempt, and we are not aware of any other analysis of the error. We should stress here that if we had used Napier's rounding, we would not have obtained $b_{50} = 9995001.224804$ but $b_{50} = 9995001.224826$, as explained by Fischer [50]. The rounded computation will be given in a future document.

[13]In § 60 of the *Constructio*, Napier observes some inconsistencies in the table and suggests a way to improve it, without noticing that at least part of the problem lies in a computation error.

The third sequence c is actually made of 69 subsequences. Each of these subsequences uses the ratio $r_3 = 0.9995$. The first of these subsequences is $c_{0,0}, c_{1,0}, c_{2,0}, \ldots, c_{20,0}$ and it starts with $c_{0,0} = 10^7$. The second subsequence is $c_{0,1}, c_{1,1}, c_{2,1}, \ldots, c_{20,1}$. The 69th subsequence is $c_{0,68}, c_{1,68}, c_{2,68}, \ldots, c_{20,68}$. For $1 \leq i \leq 20$ and $0 \leq j \leq 68$, we have $\frac{c_{i,j}}{c_{i-1,j}} = r_3$. In particular, $c_{1,0} = 9995000 \approx b_{50}$ because $0.99999^{50} \approx 0.9995$.

The relationship between one subsequence and the next one is by a constant ratio $r_4 = 0.99$ and $\frac{c_{0,j+1}}{c_{0,j}} = r_4$. Moreover, $c_{20,i} \approx c_{0,i+1}$, because $0.9995^{20} \approx 0.99$.

Napier computed the first column of the third table using 5 decimal places. Then, he used these values to compute the other values of the rows with 4 decimal places. Napier found $c_{20,0} = 9900473.57808$ [137, p. 14] (exact: 9900473.578023),[14] $c_{20,1} = 9801468.8423$ [137, p. 15] (exact: 9801468.842243), $c_{20,2} = 9703454.1539$ [137, p. 16] (exact: 9703454.153821), $\ldots$, $c_{0,68} = 5048858.8900$ [137, p. 16] (exact: 5048858.887871), $c_{20,68} = 4998609.4034$ [137, p. 16] (exact: 4998609.401853).

Napier has therefore divided the interval between 10^7 and about $5 \cdot 10^6$ into $69 \times 20 = 1380$ subintervals with a constant ratio, although there are some junctions between the 69 subsequences. Then, each of these 1380 intervals could itself be divided into 50 intervals corresponding to a ratio of 0.99999, and these intervals could in turn be divided into 100 intervals whose ratio is 0.9999999. These sequences therefore provide an approximation of a subdivision of the interval between 10^7 and $5 \cdot 10^6$ into many small intervals corresponding to the ratio 0.9999999. If the logarithm of 0.9999999 could be computed, these sequences would provide a means to compute approximations of the logarithms of the other values in the sequence.

The values of the third table are now dense enough to be used in conjunction with the sines of angles from 90° to 30°, whose sine is 5000000 when the *sinus totus* is 10^7.

Ideally, Napier would have divided the whole interval $[10^7—5 \cdot 10^6]$ using the ratio r_1, but this was an impossible task. Using the ratio 0.9999999 a total of $100 \times 50 \times 20 \times 69 = 6900000$ times, he would have obtained as his last value 5015760.517. He would in fact have needed to go a little further to be below $5 \cdot 10^6$. But if he had proceeded this way, by iterative divisions, he would have introduced important rounding errors. On the contrary, with his three sequences, Napier was able to keep the rounding errors to a minimum,

[14] Again, in this work, we consider ideal computations and we postpone the examination of rounded calculations to a separate work. Using Napier's rounding, he should actually have obtained $c_{20,0} = 9900473.57811$ and the other values of the table should also slightly be different.

since he never had more than 100 multiplications in a row, and in fact mostly only 20.

2.4 Approximation of the logarithms

Napier's next task is not to compute mere approximations of the logarithms of each value in the sequences, but interval approximations,[15] that is, bounds for each logarithm, taking in particular into account the fact that the various sequences are not seamless. By using interval approximations, Napier knows exactly the accuracy of his computations.

2.4.1 Computing the first logarithm

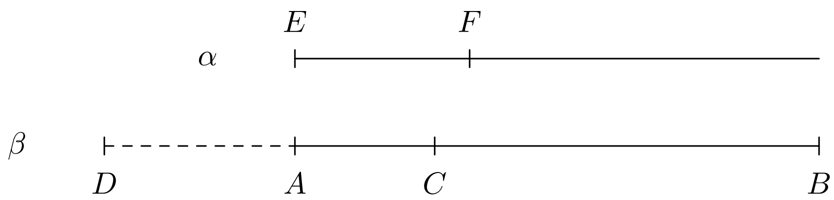


Figure 2: The bounds of $\lambda_n(CB)$.

Napier first tries to compute the logarithm of $a_1 = 9999999$. He considers the configuration of figure 2 where A and B are the endpoints of β and C is some point in between. E is the origin of line α and F is the point of α corresponding to C. The β line is extended to the left to D such that the time T taken to go from D to A is equal to the time taken to go from A to C. This, as was explained above, entails that

$$\frac{CB}{AB} = \frac{AB}{DB}. \tag{2}$$

Since EF is the logarithm of CB, and since the speed of β is equal to the distance between β and B and since this distance decreases, the speed of β also decreases, and naturally $AC < EF$. Similarly, $DA > EF$. We now have two bounds for the logarithm of CB:

$$AC < \lambda_n(CB) < DA$$

[15] What Napier actually does is interval arithmetic. This field was only developed extensively in the 20th century, starting with Young's seminal work [205]. For a popular introduction to interval arithmetic, see Hayes' article in the *American Scientist* [80].

Using equation (2), we can compute these bounds:

$$
\begin{aligned}
AC &= AB - CB = 10^7 - CB \\
DA &= DB - AB = AB\left(\frac{DB}{AB} - 1\right) = AB\left(\frac{AB}{CB} - 1\right) = AB \times \frac{AB - CB}{CB} \\
&= AB \times \frac{AC}{CB}
\end{aligned}
$$

In practice, this gives

$$10^7 - CB < \lambda_n(CB) < 10^7 \left(\frac{10^7 - CB}{CB}\right) \tag{3}$$

Taking $CB = 9999999$, we have:

$$1 < \lambda_n(9999999) < \frac{10^7}{9999999} \approx 1 + 10^{-7}$$

In the *Constructio*, Napier gives the upper bound 1.00000010000001 [137, p. 21].

Napier decided to take the average of the two bounds and set $\lambda_n(9999999) \approx 1.00000005$ [137, p. 21]. The exact value of $\lambda_n(9999999)$ is

$$10^7(\ln(10^7) - \ln(9999999)) = 1.000000050000003\ldots$$

Napier's approximation was therefore excellent.[16]

Napier can now compute all the other logarithms in the first sequence a_i, since he has approximations of the first two values of the sequence:

$$
\begin{gathered}
\lambda_n(a_0) = 0 \\
1 < \lambda_n(a_1) < \frac{10^7}{9999999}
\end{gathered}
$$

We also have $\lambda_n(a_{i+1}) - \lambda_n(a_i) = \lambda_n(a_1) - \lambda_n(a_0) = \lambda_n(a_1)$. Hence $\lambda_n(a_i) = i\lambda_n(a_1)$ and

$$i < \lambda_n(a_i) < i \times \frac{10^7}{9999999}$$

Napier takes as approximation of $\lambda_n(a_i)$:

$$\lambda_n(a_i) \approx i \times 1.00000005$$

[16] Some authors write incorrectly that Napier took the logarithm of 9999999 to be 1. This is for instance the case of Delambre [43, vol. 1, p. XXXV]. The logarithm of 9999999 may have been 1 in Napier's first experiments, but not in the actual *Descriptio*.

and fills the table in section 10.1. In order to distinguish the theoretical values of the logarithms from the values corresponding to Napier's process, we will denote the latter with $l_1(a_i)$ for the first table, $l_2(b_i)$ for the second table, and $l_3(c_{i,j})$ for the third table. Moreover, if necessary, we will use $l'_i(a_j)$ for Napier's actual table values when they differ from the recomputed ones.[17] We will always have

$$\begin{aligned}\lambda_n(a_i) &\approx l_1(a_i)\\ \lambda_n(b_i) &\approx l_2(b_i)\\ \lambda_n(c_{i,j}) &\approx l_3(c_{i,j})\end{aligned}$$

Given the excellent approximation of $\lambda_n(a_1)$, all the values of the first table are also excellent approximations. In our reconstruction, we have given the approximations Napier would have obtained, had his computations been correct and not rounded. In other words, our only approximation was to take the approximation of the first logarithm as an arithmetic mean of the bounds.

The last value of the first table is $\lambda_n(a_{100})$ and its bounds by the above reasoning are (with an exact computation):

$$100 < \lambda_n(a_{100}) < 100.000010000001\ldots$$

2.4.2 Computing the first logarithm of the second table

Next, Napier tackles the computation of the logarithm of $b_1 = 9999900$. Napier already knows the logarithm of $a_{100} \approx b_1$, but he will not consider the logarithms of a_{100} and b_1 to be equal. Instead, what Napier does is to consider the difference between $\lambda_n(b_1)$ and $\lambda_n(a_{100})$ and he shows that this difference is bounded. He proceeds as follows. First, he observes that if $\frac{a}{b} = \frac{c}{d}$, then $\lambda_n(a) - \lambda_n(b) = \lambda_n(c) - \lambda_n(d)$ which is a consequence of Napier's definition. Then, he considers figure 3. Let $AB = 10^7$ and CB and DB be the two logarithms whose difference will be considered. Points E and F are defined such that

$$\frac{EA}{AB} = \frac{CD}{DB}$$
$$\frac{AF}{AB} = \frac{CD}{CB}$$

[17] l' slightly differs from l as a consequence of Napier's errors or rounding. l differs from λ_n because Napier took the arithmetic mean when approximating the values of the logarithms.

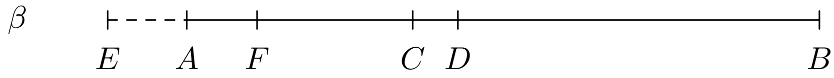


Figure 3: Bounding the difference of two logarithms.

It follows that

$$\frac{EB}{AB} = \frac{\frac{(AB)(CD)}{DB} + AB}{AB} = \frac{AB\left(\frac{CD}{DB}+1\right)}{AB} = \frac{CD+DB}{DB} = \frac{CB}{DB}$$
$$\frac{FB}{AB} = \frac{AB-FA}{AB} = \frac{AB - AB\cdot\frac{CD}{CB}}{AB} = \frac{CB-CD}{CB} = \frac{DB}{CB}$$

and therefore

$$\lambda_n(DB) - \lambda_n(CB) = \lambda_n(FB) - \lambda_n(AB) = \lambda_n(FB)$$

As seen above, since $\frac{FB}{AB} = \frac{AB}{EB}$, we have

$$AF < \lambda_n(FB) < EA$$

and therefore

$$\frac{(AB)(CD)}{CB} < \lambda_n(DB) - \lambda_n(CB) < \frac{(AB)(CD)}{DB} \tag{4}$$

Taking $CB = a_{100}$ and $DB = b_1$, we have

$$10^7 \times \frac{(a_{100}-b_1)}{a_{100}} < \lambda_n(b_1) - \lambda_n(a_{100}) < 10^7 \times \frac{(a_{100}-b_1)}{b_1}$$

which numerically gives (with the exact values)

$$0.00049500333301\ldots < \lambda_n(b_1) - \lambda_n(a_{100}) < 0.00049500333303\ldots$$

Since the two bounds were very close, Napier took $\lambda_n(b_1) - \lambda_n(a_{100}) = 0.0004950$, and from the approximation of $\lambda_n(a_{100})$ found above, he obtained [137, pp. 29–30]:

$$100 + 0.0004950\ldots < \lambda_n(a_{100}) + (\lambda_n(b_1) - \lambda_n(a_{100})) < 100.0000100\ldots + 0.0004950\ldots$$
$$100.0004950\ldots < \lambda_n(b_1) < 100.0005050\ldots$$

More accurate bounds are:

$$100.00049500333301\ldots < \lambda_n(b_1) < 100.00050500333403\ldots$$

Napier chose to take the average between the two bounds, namely $l_2(b_1) = 100.0005$, which is an excellent approximation of the real value

$$\lambda_n(b_1) = 100.00050000333\ldots$$

Napier now computes all the logarithms of the second sequence, using the approximation

$$\lambda_n(b_i) \approx i \times 100.0005$$

The bounds of the last value, $\lambda_n(b_{50})$ are

$$50 \times 100.0004950\ldots < \lambda_n(b_{50}) < 50 \times 100.0005050\ldots$$

and Napier chose $l_2(b_{50}) = 5000.0250000$ which is also an excellent approximation of the real value

$$\lambda_n(b_{50}) = 5000.0250001\ldots$$

2.4.3 Computing two logarithms of the third table

In order to complete the third table, Napier had to compute approximations of two logarithms: $\lambda_n(c_{1,0}) = \lambda_n(9995000)$ and $\lambda_n(c_{0,1}) = \lambda_n(9900000)$.

We might first think of using the fact that b_{50} is very close to $c_{1,0}$, and consider the difference $\lambda_n(c_{1,0}) - \lambda_n(b_{50})$. Its bounds can be found using equation (4):

$$10^7 \frac{b_{50} - c_{1,0}}{b_{50}} < \lambda_n(c_{1,0}) - \lambda_n(b_{50}) < 10^7 \frac{b_{50} - c_{1,0}}{c_{1,0}}$$

that is, using the *exact* value of b_{50}:

$$1.225416581\ldots < \lambda_n(c_{1,0}) - \lambda_n(b_{50}) < 1.225416731\ldots$$

But in fact, Napier obtains even better bounds, in that he goes back to the first table, which is the most dense one. Napier first computes x such that $\frac{x}{10^7} = \frac{c_{1,0}}{b_{50}}$ which gives $x = 9999998.774583418771\ldots$. Napier had actually found $x = 9999998.7764614$ [137, p. 31].[18] Then, we have $\lambda_n(c_{1,0}) - \lambda_n(b_{50}) = \lambda_n(x) - \lambda_n(10^7) = \lambda_n(x)$. We can now express the bounds of $\lambda_n(x) - \lambda_n(a_1)$, the closest value to x in the first table being $a_1 = 9999999$:

[18] The difference is essentially due to the error on b_{50}, when constructing the sequence b_i.

$$10^7\frac{(a_1-x)}{a_1} < \lambda_n(x)-\lambda_n(a_1) < 10^7\frac{(a_1-x)}{x}$$

$$0.2254166037\ldots < \lambda_n(x)-\lambda_n(a_1) < 0.2254166088\ldots$$

$$1 < \lambda_n(a_1) < 1.000000100\ldots$$

Hence

$$1.2254166037\ldots < \lambda_n(x) < 1.2254167088\ldots$$

the exact value being $1.225416656\ldots$.

These bounds are more accurate than the ones obtained using the second table only, but the first bounds were also very good.

Napier had actually found

$$1.2235386\ldots < \lambda_n(x) < 1.2235387\ldots$$

and the difference with the real bounds is a consequence of Napier's error on b_{50} which has translated in an error of about 0.002 on x.

From this, we obtain (using the exact values of the bounds)

$$5000.0247501\ldots + 1.2254166\ldots < \lambda_n(c_{1,0}) < 5000.0252501\ldots + 1.2254167\ldots$$

that is (using the exact values)

$$5001.2501667\ldots < \lambda_n(c_{1,0}) < 5001.2506668\ldots$$

The average of the bounds is 5001.2504168, but Napier found 5001.2485387 (see [137, p. 31]). We take $l_3(c_{1,0}) = 5001.2504168$.

The difference between Napier's value and the exact one is still due to the error on b_{50}. This error will propagate and increase when building the third table.

The exact value of $\lambda_n(c_{1,0})$ is $5001.25041682\ldots$ and Napier's (ideal)[19] approximation is again excellent.

Napier can then compute approximations of $c_{i,0}$ for $i \leq 20$. For $c_{20,0}$, the exact bounds are

$$100025.003335\ldots < \lambda_n(c_{20,0}) < 100025.013337\ldots$$

and Napier uses the value 100024.9707740 [137, p. 32] instead of the better $100025.008\ldots$, still as a consequence of the error on b_{50}. However, rounded to one decimal, both values are identical (see [137, p. 34]).

[19] Ideal, if Napier hadn't made any error.

Next, Napier observes that $c_{20,0} \approx c_{0,1}$. Similarly, Napier bounds the difference $\lambda_n(c_{0,1}) - \lambda_n(c_{20,0})$. If he had used the first column of the third table, he would have obtained:

$$10^7 \frac{(c_{20,0} - c_{0,1})}{c_{20,0}} < \lambda_n(c_{0,1}) - \lambda_n(c_{20,0}) < 10^7 \frac{(c_{20,0} - c_{0,1})}{c_{0,1}}$$

that is, using the *exact* value of $c_{20,0}$:

$$478.33875779\ldots < \lambda_n(c_{0,1}) - \lambda_n(c_{20,0}) < 478.36163968\ldots$$

But, what Napier did, was to use the second table instead of the first column of the third table. Napier found x such that $\frac{x}{10^7} = \frac{c_{0,1}}{c_{20,0}}$, which gives $x = 9999521.66124220\ldots$. Napier had actually found $x = 9999521.6611850$ [137, p. 33]. This value is outside of the range of the first table, but it is within the range of the second table. Then, we have $\lambda_n(c_{0,1}) - \lambda_n(c_{20,0}) = \lambda_n(x) - \lambda_n(10^7) = \lambda_n(x)$. We can now express the bounds of $\lambda_n(b_5) - \lambda_n(x)$, the closest value to x in the first table being $b_5 = 9999500.009999900\ldots$:

$$10^7 \frac{(x - b_5)}{x} < \lambda_n(b_5) - \lambda_n(x) < 10^7 \frac{(x - b_5)}{b_5}$$

$$21.6522780 < \lambda_n(b_5) - \lambda_n(x) < 21.6523249\ldots$$

$$-21.6523250 < \lambda_n(x) - \lambda_n(b_5) < -21.6522780$$

$$500.0024750 < \lambda_n(b_5) < 500.0025251$$

Hence

$$478.3501501 < \lambda_n(x) < 478.3502471$$

These bounds are much more accurate than the ones obtained using the third table only.

Napier had actually found

$$478.3502290 < \lambda_n(x) < 478.3502812.$$

From this, we obtain (using the exact values of the bounds)

$$100025.003335\ldots + 478.350150\ldots < \lambda_n(c_{0,1}) < 100025.013337\ldots + 478.350247\ldots$$

that is (using exact values)

$$100503.353485 < \lambda_n(c_{0,1}) < 100503.363585$$

The average of the two bounds is 100503.358535 and the exact value of $\lambda_n(c_{0,1})$ is 100503.35853501 .... Napier, instead, found 100503.3210291 [137, p. 33] and the difference between these values is still only due to the error on b_{50}.

Having now approximations for $\lambda_n(c_{20,0})$ and $\lambda_n(c_{0,1})$, Napier can obtain bounds for all the values in the third table. First, he obtains bounds for all $\lambda_n(c_{0,j})$, because $\lambda_n(c_{0,j}) - \lambda_n(c_{0,0}) = j \times (\lambda_n(c_{0,1}) - \lambda_n(c_{0,0}))$, hence

$$\lambda_n(c_{0,j}) = j \times \lambda_n(c_{0,1})$$

If necessary, the bounds on $\lambda_n(c_{0,1})$ can be used to find bounds for $\lambda_n(c_{0,j})$.

Finally, we have

$$\begin{aligned} \lambda_n(c_{i,j}) - \lambda_n(c_{0,j}) &= i \times (\lambda_n(c_{1,j}) - \lambda_n(c_{0,j})) \\ &= i \times (\lambda_n(c_{1,0}) - \lambda_n(c_{0,0})) \\ &= i \times \lambda_n(c_{1,0}) \end{aligned}$$

and

$$\begin{aligned} \lambda_n(c_{i,j}) &= \lambda_n(c_{0,j}) + i \times \lambda_n(c_{1,1}) \\ &= j \times \lambda_n(c_{0,1}) + i \times \lambda_n(c_{1,0}) \end{aligned}$$

All of the logarithms of the third table are computed as a mere linear combination of the logarithms $\lambda_n(c_{1,0})$ and $\lambda_n(c_{0,1})$.

2.5 Finding the logarithms of all sines

As we have shown, the sole purpose of the first and second tables was to build the third table.

Now, the third table will be used to find the logarithms of all the integers between 0 and 10^7, or rather, of all those which are sine values, the radius being taken equal to 10^7.

We will use the following notation to denote such sine values, n standing for Napier:

$$\sin_n x = 10^7 \sin x$$

At that time, the sine was not considered as a ratio, but as the length of a semi-chord in a circle of a certain radius. For Napier, the sine of 90° is 10^7.

Napier, of course, did not have an exponent notation.

2.5.1 The source of the sines

First, Napier needed the sines and cosines for every minute between 0° and 45°. Napier's *Constructio* is somewhat misleading on this matter. For those who would like to construct such a table, Napier suggests to use a table such as Reinhold's table or a more accurate one [137, p. 45]. The table he had in mind must be the one published in 1554 [151]. Reinhold's tables contain a table of tangents and a table of sine, the latter for every minute. It appears that the sine values used by Napier agree with those of Reinhold from 0° to 89°, except for a few typos. The value of the *sinus total* is also 10^7 in Reinhold's tables. However, there are a number of differences from 89° to 90°, showing that Napier necessarily used at least another different source.

It is also easy to see that Napier cannot have used Rheticus' 1551 canon [152], as it only gives the trigonometric functions every 10 minutes. He could however have used Rheticus' *Opus palatinum* (1596) [153] which gave the sines with a radius of 10^{10} every 10 seconds, and Napier would merely have had to cut off three digits and round. The comparison with Rheticus' *Opus palatinum* shows a few differences, suggesting that this was probably not Napier's source. Moreover, Napier had probably started his work before any copy of the *Opus palatinum* reached him.

The actual source was found in 1990 by Glowatzki and Göttsche who were investigating Regiomontanus' tables [70]. They concluded that Napier either used the table published by Fincke in 1583 [49], or the one published by Lansberge in 1591 [193]. We have not consulted Fincke's table, but we have found that the values are the same in Lansberge's 1591 edition[20] and in Napier's tables.

Both Fincke and Lansberge's tables go back to those of Regiomontanus (1436–1476), and perhaps even to those of Bianchini (1410–c1469) [169, p. 421].[21]

In any case, whatever the source, Napier had to obtain the logarithm for every sine. There were a total of $90 \times 60 = 5400$ different sine values and some of these values could be obtained from others, as we will see later.

Computing the initial tables and all logarithms took Napier 20 years.

[20] One should be careful, because the 1604 edition of Lansberge does not have the same sine table as the 1591 edition. Only the 1591 edition agrees with Napier.

[21] Detailed accounts of the history of trigonometric tables can be found in particular in the works of Braunmühl [200], Zeller [206], and van Brummelen [192]. See also Lüneburg's section on sine tables [106, pp. 148–162].

2.5.2 Computing a logarithm within the range of the third table

We have seen earlier, when computing the first logarithm, that

$$10^7 - n < \lambda_n(n) < 10^7 \frac{(10^7 - n)}{n}.$$

Setting $x = 10^7 - n$, Napier's procedure amounts to approximate $\lambda_n(n)$ by $\frac{1}{2}\left(x + \frac{x}{1-\frac{x}{10^7}}\right) \approx x + \frac{x^2}{2\cdot 10^7}$.

Therefore, if $\frac{x^2}{2\cdot 10^7}$ can be neglected, a good approximation of $\lambda_n(n)$ is $x = 10^7 - n$. Napier uses this approximation when $n \geq 9996700$, that is, when $\frac{x^2}{2\cdot 10^7} \leq 0.5445\ldots$. Napier chose this limit, because he found, using equation (3), that $3300 < \lambda_n(9996700) < 3301$ [137, p. 36], that is, the smallest value such that its logarithm is bounded by two consecutive integers. The correct bounds are actually

$$3300 < \lambda_n(9996700) < 3301.0893\ldots$$

and Napier was slightly wrong, but in any case, Napier takes the lower bound as an approximation of $\lambda_n(n)$ when $n \geq 9996700$, because he considers that the error will then be less than 0.5. An exact computation shows that the error is smaller than 0.5 only when $n > 9996837$. But even if this computation was slightly incorrect (assuming that the limit of this case in the *Constructio* is the limit used in the construction of the table), the problem actually only concerns two sine values of the table of log. sines, namely those of $88°32'$ and $88°33'$.

When $5 \cdot 10^6 \leq n < 9996700$ and n is not in the third table, Napier proceeds as follows. First, i, j are found such that $c_{i+1,j} < n < c_{i,j}$ or such that $c_{0,j+1} < n < c_{20,j}$. We then have two numbers x, y, such that $x < n < y$ and we know $\lambda_n(x)$ and $\lambda_n(y)$.

We can now bound either $\lambda_n(x) - \lambda_n(n)$ or $\lambda_n(n) - \lambda_n(y)$. In the former case

$$-10^7\left(\frac{n-x}{x}\right) < \lambda_n(n) - \lambda_n(x) < -10^7\left(\frac{n-x}{n}\right)$$

and in the latter case

$$10^7\left(\frac{y-n}{y}\right) < \lambda_n(n) - \lambda_n(y) < 10^7\left(\frac{y-n}{n}\right)$$

We also have bounds for $\lambda_n(x)$ and $\lambda_n(y)$ and by adding them, we would obtain bounds for $\lambda_n(n)$. We can however forget about the bounds and use only approximations of $\lambda_n(n) - \lambda_n(x)$ or $\lambda_n(n) - \lambda_n(y)$ and of $\lambda_n(x)$ or $\lambda_n(y)$.

For example, if $n = 5 \cdot 10^6$, we can take $x = c_{20,68} = 4998609.401853$, $y = c_{19,68} = 5001109.956832$ and find bounds for $\lambda_n(n) - \lambda_n(y)$, since n is closest to y. We have $\lambda_n(y) = 6929252.1$. Therefore,

$$10^7 \frac{1109.956832}{5001109.956832} < \lambda_n(n) - \lambda_n(y) < 10^7 \frac{1109.956832}{5 \cdot 10^6}$$

$$2219.420\ldots < \lambda_n(n) - \lambda_n(y) < 2219.913\ldots$$

$$\lambda_n(n) \approx 6929252.1 + 2219.7 = 6931471.8$$

If $n = 8 \cdot 10^6$, we can take $x = c_{5,22} = 7996285.161399$, $y = c_{4,22} = 8000285.304051$ and find bounds for $\lambda_n(n) - \lambda_n(y)$, since n is closest to y. We have $\lambda_n(y) = 2231078.9$. Therefore,

$$10^7 \frac{285.304051}{8000285.304051} < \lambda_n(n) - \lambda_n(y) < 10^7 \frac{285.304051}{8 \cdot 10^6}$$

$$356.61\ldots < \lambda_n(n) - \lambda_n(y) < 356.63\ldots$$

$$\lambda_n(n) \approx 2231078.9 + 356.6 = 2231435.5$$

Napier actually found $\lambda_n(5000000) = 6931469.22$ and $\lambda_n(8000000) = 2231434.68$. The exact values are 6931471.805599 and 2231435.513142, but as the above computations show, Napier could have found much more accurate values, had he not made an error in his second table and had he computed correctly the first logarithms of the third table.

2.5.3 Napier's short table

In his *Constructio* [134, 137], Napier gives a table which sums up the values of the differences of the logarithms of two numbers, if these numbers correspond to a given ratio. The table will be used to find the logarithms of numbers outside the range $[10^7 — 5 \cdot 10^6]$. Napier had shown earlier that if $\frac{a}{b} = \frac{c}{d}$, then $\lambda_n(a) - \lambda_n(b) = \lambda_n(c) - \lambda_n(d)$. The difference $\lambda_n(p) - \lambda_n(q)$ therefore only depends on $\frac{p}{q}$, not on the actual values of p and q. It is also important to realize that $\lambda_n(p) - \lambda_n(q) = -10^7 \ln(p/q) = \lambda_n(p/q) - 10^7 \ln(10^7)$ and therefore that $\lambda_n(p) - \lambda_n(q) \neq \lambda_n(p/q)$.

We now set $\rho(p/q) = \lambda_n(q) - \lambda_n(p)$. In order to fill his table, Napier uses the value of $\lambda_n(5000000)$ computed earlier. He then obtains $\rho(2) = \lambda_n(5000000) - \lambda_n(10000000) = \lambda_n(5000000) = 6931469.22$. This is the difference corresponding to a ratio of 2. Then, Napier computes $\lambda_n(x/4) - \lambda_n(x/2) = \lambda_n(x/2) - \lambda_n(x) = \rho(2)$ hence $\rho(4) = \lambda_n(x/4) - \lambda_n(x) = 2(\lambda_n(x/2) - \lambda_n(x)) = 2\rho(2)$. Similarly, he obtains $\rho(8) = 3\rho(2) = 20794407.66$.

Then, Napier uses the value of $\lambda_n(8000000)$, also computed earlier, and writes $\lambda_n(1000000) - \lambda_n(8000000) = \rho(8)$, hence

$$\lambda_n(1000000) = \lambda_n(8000000) + \rho(8) = 23025842.34.$$

Finally, $\rho(10) = \lambda_n(1000000) - \lambda_n(10000000) = \lambda_n(1000000)$. By iterating this process, Napier computes $\rho(20)$, $\rho(40)$, etc., until $\rho(10000000)$. In general, we have $\rho(ab) = \rho(a) + \rho(b)$ and $\rho(a^b) = b\rho(a)$.

Napier can now complete his short table (figure 4).

It can be noticed that $\rho(10) = 10^7 \ln 10$, and so, in a way, Napier accidentally computed $\ln 10$, but also $\ln 2$, $\ln 4$, $\ln 8$, etc. How accurate his calculations were can be judged by the accuracy of $\ln 2$ and $\ln 10$ which have respectively five and six correct places.

Some authors have wondered about the value of $\lambda_n(1)$.[22] The short table readily gives an approximation of $\lambda_n(1)$, since $\lambda_n(1) = \lambda_n(1) - \lambda_n(10^7) = \rho(10^7)$, but in Napier's table the small differences at the beginning of the table accumulate and produce greater differences for $\lambda_n(1)$. Using the modern expression seen previously, we find of course $\lambda_n(1) = 161180956.509\ldots$.

Using our table, we find therefore $\lambda_n(1) = 161180955.81$, but Napier had $\lambda_n(1) = 161180896.38$ [137, p. 39]. However, this value was of little importance in the construction of his logarithms.

2.5.4 Computing a logarithm outside the range of the third table

If $n < 5{\cdot}10^6$, we can determine $k = 2, 4, 8, 10, \ldots$, such that $5{\cdot}10^6 \le kn \le 10^7$. There are at most two such values, and we can choose the one we like. Then, we know that $\lambda_n(n) - \lambda_n(kn) = \rho(k)$ and $\lambda_n(n) = \lambda_n(kn) + \rho(k)$.

For example, let us compute $\lambda_n(87265)$, where 87265 is an approximation of $\sin_n 30'$. The exact value is $\sin_n 30' = 87265.354\ldots$ In our case, we can take $k = 100$. We first find an approximation of $\lambda_n(8726500)$ using the technique seen above. Setting $n = 8726500$, we have:

$$c_{12,13} < n < c_{11,13}$$

with $c_{11,13} = 8727067.052058$, $c_{12,13} = 8722703.518532$, and $\lambda_n(c_{11,13}) =$

[22]Bower shows for instance how $\lambda_n(1)$ can be computed using the methods set forth in the 1614 and 1616 editions of the *Descriptio*. Because of Napier's slight errors in his tables, and because of other approximations, the use of Napier's original tables would give a slightly different value, as explained by Bower [18]. See also Sommerville [179].

Given Proportions of Sines. i		Corresponding Differences of Logarithm. d_i	Given Proportions of Sines. i		Corresponding Differences of Logarithm. d_i
2	to one	6931469.22	8000	to one	89871934.68
4	”	13862938.44	10000	”	92103369.36
8	”	20794407.66	20000	”	99034838.58
10	”	23025842.34	40000	”	105966307.80
20	”	29957311.56	80000	”	112897777.02
40	”	36888780.78	100000	”	115129211.70
80	”	43820250.00	200000	”	122060680.92
100	”	46051684.68	400000	”	128992150.14
200	”	52983153.90	800000	”	135923619.36
400	”	59914623.12	1000000	”	138155054.04
800	”	66846092.34	2000000	”	145086523.26
1000	”	69077527.02	4000000	”	152017992.48
2000	”	76008996.24	8000000	”	158949461.70
4000	”	82940465.46	10000000	”	161180896.38

Given Proportions of Sines. i		Corresponding Differences of Logarithm. d_i	Given Proportions of Sines. i		Corresponding Differences of Logarithm. d_i
2	to one	6931471.77	8000	to one	89871967.80
4	”	13862943.54	10000	”	92103403.32
8	”	20794415.31	20000	”	99034875.09
10	”	23025850.83	40000	”	105966346.86
20	”	29957322.60	80000	”	112897818.63
40	”	36888794.37	100000	”	115129254.15
80	”	43820266.14	200000	”	122060725.92
100	”	46051701.66	400000	”	128992197.69
200	”	52983173.43	800000	”	135923669.46
400	”	59914645.20	1000000	”	138155104.98
800	”	66846116.97	2000000	”	145086576.75
1000	”	69077552.49	4000000	”	152018048.52
2000	”	76009024.26	8000000	”	158949520.29
4000	”	82940496.03	10000000	”	161180955.81

Figure 4: Napier's short table (above) and with ideal values (below). The first table was copied from the *Constructio*, whereas the second is the table Napier should have obtained, had he made no error and no rounding. This table differs from the exact values (figure 5), because Napier takes the average of the bounds.

Given Proportions of Sines. i		Corresponding Differences of Logarithm. d_i	Given Proportions of Sines. i		Corresponding Differences of Logarithm. d_i
2	to one	6931471.81	8000	to one	89871968.21
4	”	13862943.61	10000	”	92103403.72
8	”	20794415.42	20000	”	99034875.53
10	”	23025850.93	40000	”	105966347.33
20	”	29957322.74	80000	”	112897819.14
40	”	36888794.54	100000	”	115129254.65
80	”	43820266.35	200000	”	122060726.46
100	”	46051701.86	400000	”	128992198.26
200	”	52983173.67	800000	”	135923670.07
400	”	59914645.47	1000000	”	138155105.58
800	”	66846117.28	2000000	”	145086577.39
1000	”	69077552.79	4000000	”	152018049.19
2000	”	76009024.60	8000000	”	158949521.00
4000	”	82940496.40	10000000	”	161180956.51

Figure 5: Exact values of the ratios, rounded to two places.

1361557.4. Therefore

$$\begin{aligned}\lambda_n(n) &\approx \lambda_n(c_{11,13}) + 10^7 \left(\frac{c_{11,13} - n}{n}\right) \\ &\approx 1361557.4 + 649.80 \\ &\approx 1362207.20\end{aligned}$$

Then

$$\begin{aligned}\lambda_n(87265) &= \lambda_n(100 \times 87265) + \rho(100) \\ &\approx 1362207.20 + 46051701.66 \\ &\approx 47413908.86\end{aligned}$$

Napier found $\lambda_n(87265) = 47413852$, but he did not compute $\lambda_n(87265)$ that way. However, if he had used this method, he would still have had an error, firstly as a consequemce of the incorrect value of $c_{11,13}$ in his third table, itself as a consequence of the error on b_{50}, and secondly because of the error on $\rho(100)$. Moreover, Napier would have had to do 5400 such computations.

It is however easier to compute some of the logarithms from other logarithms. Napier actually used the following equation [137, p. 42]:

$$\lambda_n\left(\frac{1}{2}\sin_n 90\right) + \lambda_n(\sin_n \alpha) = \lambda_n\left(\sin_n \frac{\alpha}{2}\right) + \lambda_n\left(\sin_n\left(90 - \frac{\alpha}{2}\right)\right).$$

which is an immediate consequence of $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$.

With this formula, it is sufficient to compute the logarithms of the sines between 45° and 90° and therefore to make use of only half of the third table. The previous equation also yields

$$\lambda_n(\sin_n \alpha) = \lambda_n\left(\frac{1}{2}\sin_n 90\right) + \lambda_n(\sin_n 2\alpha) - \lambda_n(\sin_n(90-\alpha)) \qquad (5)$$

and if $22°30' \leq \alpha < 45°$, $\lambda_n(\sin_n \alpha)$ is expressed using already known values. The same process can be iterated, and once the values of the logarithms of $\sin_n \alpha$ for $\alpha \geq 22°30'$ are known, the values of the logarithms of $\sin_n \alpha$ for $11°15' \leq \alpha < 22°30'$ can be obtained, and so on [137, p. 43].

As an example, take $\alpha = 12°30'$. Using equation (5), we find

$$\begin{aligned}\lambda_n(\sin_n 12°30') &= \lambda_n\left(\frac{1}{2}\sin_n 90\right) + \lambda_n(\sin_n 25°) - \lambda_n(\sin_n(77°30')) \\ &= \lambda_n(5000000) + \lambda_n(4226183) - \lambda_n(9762960) \\ &= 6931469 + 8612856 - 239895 \\ &= 15304430\end{aligned}$$

and this is the value in Napier's table. In general, however, the resulting value is not exact, because all the errors add up. The first approach, in which a value is transposed to a standard interval, would be better, assuming the basic tables have been correctly computed, and the values of ρ are known with sufficient accuracy.

Figure 6 shows a more complete computation, for $\lambda_n(\sin_n 30')$ which can be obtained from values computed earlier. From this calculation, it appears that only some of the results coincide with those of Napier. For instance, using equation (5), Napier should have found $\lambda_n(\sin_n 32°) = 6350304$ and not 6350305. Three out of seven computations show a discrepancy, if we start with other values of the tables, and Napier's value of $\lambda_n(5000000)$. The most likely explanation is that Napier did the computation with more decimal places, but rounded the results.

3 Napier's 1614 tables

Napier's tables span 90 pages, one page covering half a degree (figure 7). The sines are arranged semi-quadrantically and on the same line we have (approximations of) $\sin_n(\alpha)$, $\lambda_n(\sin \alpha)$, $\lambda_n(\sin \alpha) - \lambda_n(\sin(90-\alpha))$, $\lambda_n(\sin(90-\alpha))$, $\sin(90-\alpha)$, in that order.

For instance, for $\alpha = 0°30'$, we have:

$$
\begin{aligned}
\lambda_n(\sin_n 32°) &= \lambda_n\left(\frac{1}{2}\sin_n 90\right) + \lambda_n(\sin_n 64°) - \lambda_n(\sin_n(58°)) \\
&= \lambda_n(5000000) + \lambda_n(8987946) - \lambda_n(8480481) \\
&= 6931469 + 1067014 - 1648179 \\
&= 6350304 \text{ (table: 6350305)} \\
\lambda_n(\sin_n 16°) &= \lambda_n\left(\frac{1}{2}\sin_n 90\right) + \lambda_n(\sin_n 32°) - \lambda_n(\sin_n(74°)) \\
&= \lambda_n(5000000) + \lambda_n(\sin_n 32°) - \lambda_n(9612617) \\
&= 6931469 + 6350305 - 395086 \\
&= 12886688 \text{ (table: 12886689)} \\
\lambda_n(\sin_n 8°) &= \lambda_n\left(\frac{1}{2}\sin_n 90\right) + \lambda_n(\sin_n 16°) - \lambda_n(\sin_n(82°)) \\
&= \lambda_n(5000000) + \lambda_n(\sin_n 16°) - \lambda_n(9902681) \\
&= 6931469 + 12886689 - 97796 \\
&= 19720362 \text{ (table: 19720362)} \\
\lambda_n(\sin_n 4°) &= \lambda_n\left(\frac{1}{2}\sin_n 90\right) + \lambda_n(\sin_n 8°) - \lambda_n(\sin_n(86°)) \\
&= \lambda_n(5000000) + \lambda_n(\sin_n 8°) - \lambda_n(9975640) \\
&= 6931469 + 19720362 - 24390 \\
&= 26627441 \text{ (table: 26627442)} \\
\lambda_n(\sin_n 2°) &= \lambda_n\left(\frac{1}{2}\sin_n 90\right) + \lambda_n(\sin_n 4°) - \lambda_n(\sin_n(88°)) \\
&= \lambda_n(5000000) + \lambda_n(\sin_n 4°) - \lambda_n(9993908) \\
&= 6931469 + 26627442 - 6094 \\
&= 33552817 \text{ (table: 33552817)} \\
\lambda_n(\sin_n 1°) &= \lambda_n\left(\frac{1}{2}\sin_n 90\right) + \lambda_n(\sin_n 2°) - \lambda_n(\sin_n(89°)) \\
&= \lambda_n(5000000) + \lambda_n(\sin_n 2°) - \lambda_n(9998477) \\
&= 6931469 + 33552817 - 1523 \\
&= 40482763 \text{ (table: 40482764)} \\
\lambda_n(\sin_n 30') &= \lambda_n\left(\frac{1}{2}\sin_n 90\right) + \lambda_n(\sin_n 1°) - \lambda_n(\sin_n(89°30')) \\
&= \lambda_n(5000000) + \lambda_n(\sin_n 1°) - \lambda_n(9999619) \\
&= 6931469 + 40482764 - 381 \\
&= 47413852 \text{ (table: 47413852)}
\end{aligned}
$$

Figure 6: Computation of $\lambda_n(\sin_n 30')$ using values computed previously. In each case, the table values and not the exact values have been used to compute a new value.

$$\begin{aligned}\sin_n(\alpha) &= 87265\\ \lambda_n(\sin_n\alpha) &= 47413852\\ \lambda_n(\sin_n\alpha)-\lambda_n(\sin_n(90-\alpha)) &= 47413471\\ \lambda_n(\sin_n(90-\alpha)) &= 381\\ \sin_n(90-\alpha) &= 9999619\end{aligned}$$

If we define $\sigma_n(x) = \lambda_n(\sin_n x)$ and $\tau_n(x) = \textit{differential} = \sigma_n(x)-\sigma_n(90-x)$, $\tau_n(x)$ then has a very simple expression:

$$\begin{aligned}\tau_n(x) &= \lambda_n(\sin_n x) - \lambda_n(\cos_n x)\\ &= 10^7(\ln(10^7)-\ln(\sin_n x)) - 10^7(\ln(10^7)-\ln(\cos_n x))\\ &= -10^7\ln(\tan_n x)\end{aligned}$$

τ_n is positive from 0° to 45° and negative afterwards, which is indicated by the '+|−' signs at the top of the middle column.

4 Wright's 1616 tables

When he translated the *Descriptio*, Wright has actually reset the tables [133]. The most conspicuous change is probably the translation of the heading (*Gr.* becoming *Deg.*), but in fact Wright reduced all the sines and logarithms by one figure [18, p. 14]. For instance, for 0°30′, Napier had $\sin_n(0°30') = 87265$ and $\lambda_n(\sin_n(0°30')) = 47413852$ (figure 7), whereas Wright had $\sin_w(0°30') = 8726$ and $\lambda_w(\sin_w(0°30')) = 4741385$ (figure 8). According to Oughtred, this change is made to make the interpolation easier [68, pp. 179]. This feature was reproduced in our reconstruction of the 1616 table.

Another difference was the use of a decimal point in a portion of the table (figure 8).

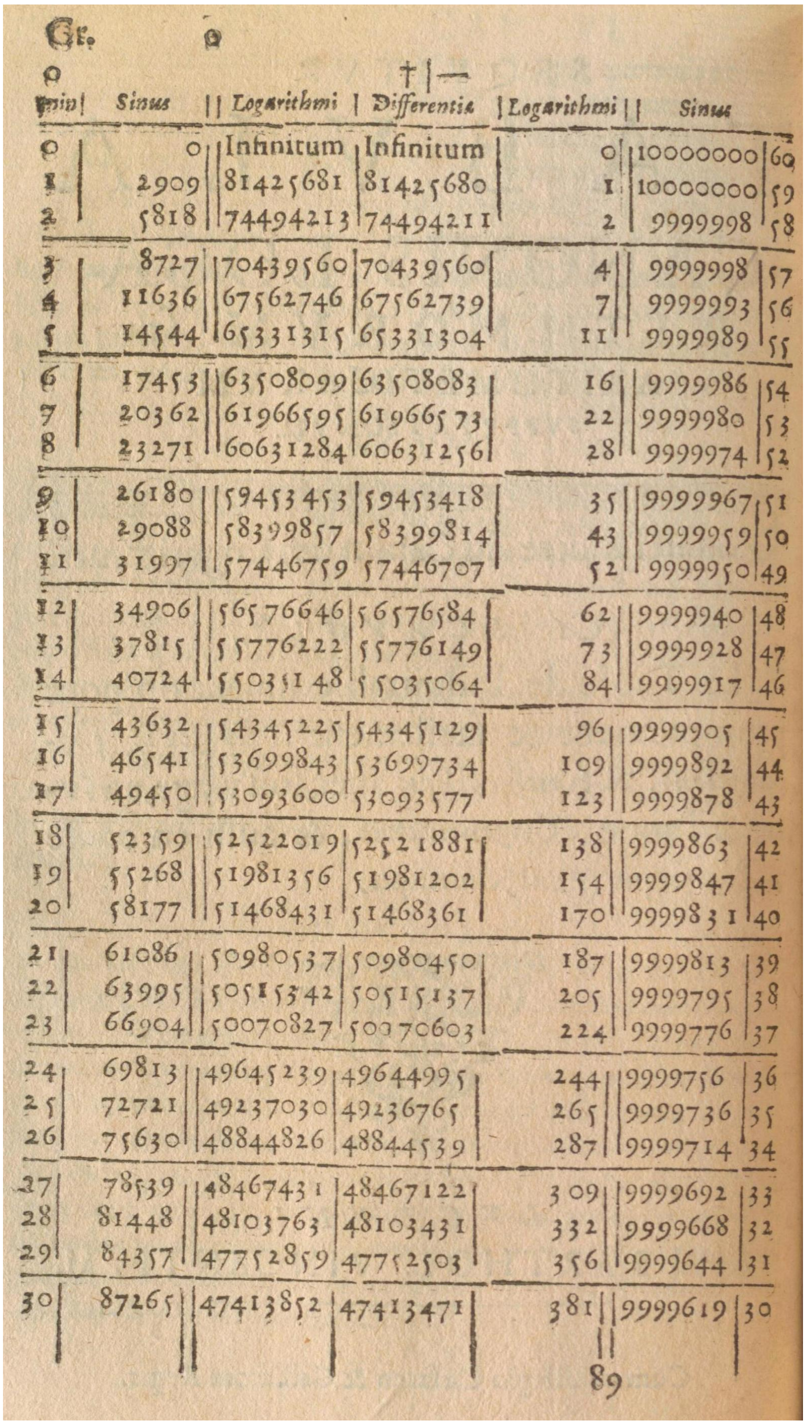
The change introduced by Wright naturally also changes the modern expression of the logarithm in these tables. By definition, we have:

$$\lambda_w(x) = \frac{1}{10}\lambda_n(10x) = 10^6(\ln(10^7)-\ln(10x))$$

and in particular

$$\lambda_w(1) = 13815510.557\ldots$$

In his article, Bower shows how values such as $\lambda_n(1)$ and $\lambda_w(1)$ can be computed using the methods set forth in Napier's *descriptio* or in its translation [18, pp. 15–16].



Gr. 0

0

min	Sinus	Logarithmi	Differentia	Logarithmi	Sinus	
0	0	Infinitum	Infinitum	0	10000000	60
1	2909	81425681	81425680	1	10000000	59
2	5818	74494213	74494211	2	9999998	58
3	8727	70439560	70439560	4	9999998	57
4	11636	67562746	67562739	7	9999993	56
5	14544	65331315	65331304	11	9999989	55
6	17453	63508099	63508083	16	9999986	54
7	20362	61966595	61966573	22	9999980	53
8	23271	60631284	60631256	28	9999974	52
9	26180	59453453	59453418	35	9999967	51
10	29088	58399857	58399814	43	9999959	50
11	31997	57446759	57446707	52	9999950	49
12	34906	56576646	56576584	62	9999940	48
13	37815	55776222	55776149	73	9999928	47
14	40724	55035148	55035064	84	9999917	46
15	43632	54345225	54345129	96	9999905	45
16	46541	53699843	53699734	109	9999892	44
17	49450	53093600	53093577	123	9999878	43
18	52359	52522019	52521881	138	9999863	42
19	55268	51981356	51981202	154	9999847	41
20	58177	51468431	51468361	170	9999831	40
21	61086	50980537	50980450	187	9999813	39
22	63995	50515342	50515137	205	9999795	38
23	66904	50070827	50070603	224	9999776	37
24	69813	49645239	49644995	244	9999756	36
25	72721	49237030	49236765	265	9999736	35
26	75630	48844826	48844539	287	9999714	34
27	78539	48467431	48467122	309	9999692	33
28	81448	48103763	48103431	332	9999668	32
29	84357	47752859	47752503	356	9999644	31
30	87265	47413852	47413471	381	9999619	30

89

Figure 7: Excerpt of Napier's table, from the 1620 reprint. This table is almost identical to the 1614 table, but the current table was reset. Compare this page with Wright's version (figure 8).

Deg. 0 + | —

mi	Sines	Logarith	Differen.	Logarith	Sines	
0	0	Infinite.	Infinite.	.0	1000000.0	60
1	291	8142567	8142568	.1	1000000.0	59
2	582	7449419	7449421	.2	999999.8	58
3	873	7043952	7043956	.4	999999.6	57
4	1164	6756275	6756274	.7	999999.3	56
5	1454	6533131	6533130	1.1	999998.9	55
6	1745	6350810	6350808	1.6	999998.6	54
7	2036	6196659	6196657	2.2	999998.0	53
8	2327	6063128	6063126	2.8	999997.4	52
9	2618	5945345	5945342	3.5	999996.7	51
10	2909	5839986	5839814	4.3	999995.9	50
11	3280	5744676	5744671	5.2	999995.0	49
12	3491	5657665	5657658	6.2	999994.0	48
13	3781	5577622	5577615	7.3	999992.8	47
14	4072	5503514	5503506	8.4	999991.7	46
15	4363	5434522	5434513	9.6	999990.5	45
16	4654	5369984	5369973	10.9	999989.2	44
17	4945	5309360	5309348	12.3	999987.8	43
18	5236	5252202	5252188	13.8	999986.3	42
19	5527	5198136	5198120	15.4	999984.7	41
20	5818	5146843	5146836	17.0	999983.1	40
21	6109	5098054	5098045	18.7	999981.3	39
22	6399	5051534	5051514	20.5	999979.5	38
23	6690	5007083	5007060	22.4	999977.6	37
24	6981	4964524	4964499	24.4	999975.6	36
25	7272	4923703	4923676	26.5	999973.6	35
26	7563	4884483	4884454	28.7	999971.4	34
27	7854	4846743	4846712	30.9	999969.2	33
28	8145	4810376	4810343	33.2	999966.8	32
29	8436	4775286	4775250	35.0	999964.4	31
30	8726	4741385	4741347	38.1	999961.9	30
						Min.

Deg. 89

Figure 8: Excerpt of Wright's translation. Compare this page with the original version (figure 7).

The expression of the differential remains similar:

$$\begin{aligned}\tau_w(x) &= \lambda_w(\sin_w x) - \lambda_w(\cos_w x) \\ &= 10^6(\ln(10\cos_w x) - \ln(10\sin_w x)) = -10^6 \ln\tan_w x\end{aligned}$$

and therefore the 1614 values of the differential could also be divided by 10. However, this process seems to have erred for the first values of the table.[23]

The second edition of Wright's translation, published in 1618, also contained an anonymous appendix, probably written by William Oughtred, where for the first time the so-called radix method for computing logarithms was used [68]. This method was described in more detail by Briggs in his *Arithmetica logarithmica* [21].

5 Decimal fractions

Decimal fractions as we know them today are used by Napier, but they were a very recent introduction. Stevin made a decisive step forward in 1585 in his work *De Thiende* [35, pp. 615–617], [170]. Fractions had been considered before, and so had the concept of position. Several mathematicians either came close to it,[24] or even propounded or used a decimal fraction notation,[25] but Stevin was the first to devote a book to that specific matter and to introduce a notation in which the fractional digits were at the same level as the integer digits, and in which a specific notation indicated the weight of each fractional digit. Stevin therefore had an (admittedly clumsy) index notation for the fractional digits, although he did not use a full index notation for the integer part. The integer part was considered as having index 0, and this 0 could be viewed as a position marker equivalent to our fractional dot. Stevin wrote 318⓪9①3②7③ for our 318.937. Stevin extended the four fundamental operations to these numbers and proved the validity of his rules. The very late introduction of these concepts, although a position concept had existed in Sumerian mathematics and decimal numeration can be traced back to Egypt in the fourth millenium B.C., is partly due to the resistance to the introduction of so-called Hindu-Arabic numerals, which were not very popular before 1500.

[23]For instance, Wright gives $\lambda_w(\sin_w 2') = 7449419$, $\lambda_w(\sin_w 89°58') = .2$ and the *differential* as 7449421. The three first lines of the table show such idiosyncrasies, but perhaps they are only printing errors.

[24]One of the earliest example is John of Murs in his *Quadripartitum numerorum* completed in 1343.

[25]For instance, Viète used decimal fractions in his *Universalium Inspectionum* (1579).

Dots had been used in the notation of numbers before 1500, but these dots were not decimal dots. They were usually used to separate groups of digits. Some writers have also written two numbers next to each other, one for the integer part and one for the fractional part, with some separating symbol (for instance '|'), but without grasping the full significance of this juxtaposition. Clavius, for instance, used a point as decimal separator in 1593, but still wrote decimal fractions as common fractions in 1608, and his grasp of the decimal notation is open to doubt [170, p. 177]. Jost Bürgi, however, put a small zero under the last integral figure in his unpublished *Coss* (called the *Arithmetica* by Cantor) completed around the end of the 16th century, and Kepler ascribed the new kind of decimal notation to Bürgi. Pitiscus also used decimal points in his *Trigonometria* published in 1608 and 1612, however, his use of them lacked consistency.[26] It was occasional, not systematic [170, p. 181].

Napier was in fact the main introducer of decimal fractions into common practice and his new mathematical instrument became the best vehicle of the decimal idea. The *Descriptio* actually only contains rare instances of decimal fractions. There are no decimal fractions in Napier's 1614 table,[27] but within the tables of Wright's 1616 translation, decimal fractions occur for angles between 89 and 90 degrees.[28] For instance, the sine of $89°30'$ degrees is given as 999961.9 and its logarithm as 38.1 (figure 8). But decimal fractions appear in full force in the *Constructio* which was written a number of years before the *Descriptio* and subtends it. Several examples of values found in the *Constructio*, and in particular of the values contained in the three tables of progressions and the short table, have already been given earlier in this document.

And one of the first sentences of the *Constructio* is the following [137, p. 8]: "In numbers distinguished thus by a period in their midst, whatever is written after the period is a fraction, the denominator of which is unity with as many cyphers after it as there are figures after the period."

[26]Cantor was one of the authors who wrote that Pitiscus' 1608 and 1612 tables exhibited the use of the dot as a separator of a decimal part [35, pp. 617–619]. This, however, is not true. Even a cursory examination of these two tables shows that they contain indeed dots, but these dots are separating groups of digits, and if some of them separate the decimal part, they can of course not be ascribed that sole meaning.

[27]It should be noted that Maseres' reprint of the *Descriptio* of course retypeset it, and the layout is slightly different. Maseres' main change was to group the figures by the introduction of commas, which do not appear in the original version [111]. Maseres gives for instance $\sin_m 43°30' = 6{,}883{,}546$ and $\lambda_m(\sin_m 43°30') = 7{,}734{,}510$.

[28]Wright must have used decimal points only when the logarithms were too small, and when he felt that rounding the values to an integer would produce a too great loss of accuracy.

6 Computing with Napier's logarithms

6.1 Basic computations

Napier's logarithms can be used for basic multiplications and divisions, which are transformed into additions and subtractions. It is easy to see that

$$\begin{aligned}
\lambda_n(ab) &= 10^7(\ln(10^7) - \ln a - \ln b) \\
&= \lambda_n(a) + \lambda_n(b) - 10^7 \ln(10^7) \\
&= \lambda_n(a) + \lambda_n(b) - \lambda_n(1) \\
\lambda_n(a/b) &= 10^7(\ln(10^7) - \ln a + \ln b) \\
&= \lambda_n(a) - \lambda_n(b) + 10^7 \ln(10^7) \\
&= \lambda_n(a) - \lambda_n(b) + \lambda_n(1)
\end{aligned}$$

The previous computations normally involve the addition or subtraction of a large number, $\lambda_n(1) = 161180896.38$ in Napier's *Constructio*.

However, in many cases, one can dispense with this constant. For instance, if we wish to compute $\lambda_n(ab/c)$, we have

$$\begin{aligned}
\lambda_n(ab/c) &= \lambda_n(ab) - \lambda_n(c) + \lambda_n(1) \\
&= \lambda_n(a) + \lambda_n(b) - \lambda_n(1) - \lambda_n(c) + \lambda_n(1) \\
&= \lambda_n(a) + \lambda_n(b) - \lambda_n(c)
\end{aligned}$$

These equations are easily transposed to Wright's version of the logarithms:

$$\begin{aligned}
\lambda_w(ab) &= \lambda_w(a) + \lambda_w(b) - \lambda_w(1) \\
\lambda_w(a/b) &= \lambda_w(a) - \lambda_w(b) + \lambda_w(1) \\
\lambda_w(ab/c) &= \lambda_w(a) + \lambda_w(b) - \lambda_w(c)
\end{aligned}$$

It may come as a surprise, however, that Napier did not give such simple examples in the *Descriptio*. Napier's simplest examples appear in chapter 5, at the end of the Book I of the *Descriptio*. He considers four cases. In the first case, a, b, and c are such that $\frac{c}{b} = \frac{b}{a}$, the values of a and b are given and c is sought. Napier shows that $\lambda_n(c) = 2\lambda_n(b) - \lambda_n(a)$, from which c is easily obtained. In the second case, we have the same proportions $\frac{c}{b} = \frac{b}{a}$, but a and c are given, and b is sought. Napier explains that the square root $b = \sqrt{ac}$ is replaced by a division by two: $\lambda_n(b) = \frac{1}{2}(\lambda_n(a) + \lambda_n(c))$. In the third case, four numbers a, b, c, and d are considered such that $\frac{b}{a} = \frac{c}{b} = \frac{d}{c}$. Knowing the first three, the fourth is sought. We have $\lambda_n(d) = \lambda_n(b) + \lambda_n(c) - \lambda_n(a)$. In the fourth case, we have the same proportions as in the third case, but a and d are given, and b and c are sought. We have $\lambda_n(c) = \lambda_n(d) + \frac{1}{3}(\lambda_n(a) - \lambda_n(d))$ and $\lambda_n(b) = \lambda_n(a) + \frac{1}{3}(\lambda_n(d) - \lambda_n(a))$.

6.2 Oughtred's radix method

In the appendix of the second edition of Wright's translation, probably authored by William Oughtred, the radix method is given for calculating the logarithm of any number. This method actually involves an extension of Napier's short table. The values in the new table were the values of $\rho_w(1)$, $\rho_w(2)$, $\rho_w(3)$, etc., which are the equivalent to $\rho(1)$, $\rho(2)$, $\rho(3)$, etc., but with one digit less.

In order to compute $\lambda_w(n)$, Oughtred's idea is to find a such that $980000 < an < 1000000$ (Wright's translation has $\lambda_w(10^6) = 0$) and such that a is a product of simple factors such as 1, 2, 3, ..., 10, 20, ..., 90, 100, ..., 1.1, 1.2, 1.3, ..., 1.01, 1.02, ..., 1.09. Then, $\lambda_w(n) = \lambda_w(an) + \rho_w(a)$ and $\rho_w(a)$ can be computed using Oughtred's table and the relation $\rho_w(ab) = \rho_w(a) + \rho_w(b)$. $\lambda_w(an)$ can be computed either by sight, when an is near 1000000, by using the closest value in the table, or by interpolation. Interestingly, the appendix also makes use of Oughtred's $\times$ for the multiplication, and abreviations for the sine, tangent, cosine, cotangent, which appear here for one of the first time in formulæ. A full analysis of the appendix was given by Glaisher [68] and we give more details on the radix method in our study of Briggs' *Arithmetica logarithmica* [161].

6.3 Negative numbers

In the *Descriptio* (book 1, chapter 1), Napier explains that numbers greater than the *sinus total* have a negative logarithm: *the Logarithmes of numbers greater then the whole sine, are lesse then nothing* (...) *the Logarithmes which are lesse then nothing, we cal Defective, or wanting, setting this marke − before them* [133, p. 6].

6.4 Scaling notation

In the *Descriptio*, Napier introduced a primitive exponent notation for manipulating numbers that need to be scaled to fit within his table. For instance, if the logarithm of 137 is sought, he finds 1371564 among the sines. The value in the table is 19866327 (exact: 19866333.98) and Napier writes 19866327−0000, meaning that four figures must be removed from the sine. Napier then manipulates such logarithms which he calls "impure," as if the number of zeros were exponents. Logarithms without a $-000...$ or $+000\ldots$ are called "pure." For Napier, the impure logarithm 23025842+0 is in fact equal to 0, and it can be added to an impure logarithm ..−0... in order to render it pure.

What Napier really does is merely to move the exponent to the mantissa. Napier is manipulating a "decimal characteristic," but transposed in his logarithms. Apart from Glaisher in 1920 [69, p. 165], Napier's notation apparently hasn't much been described, yet it is very easy to understand if we transpose it in modern notations. If x is a pure logarithm, then $x+\underbrace{000\ldots...0}_{n}$ is merely a notation for $x - n\rho(10)$. And $x-\underbrace{000\ldots...0}_{n}$ is a notation for $x + n\rho(10)$. Given that $\rho(10) = \lambda_n(1) - \lambda_n(10)$, it is very easy to see that $\lambda_n(a) = \lambda_n(10^n a) + n\rho(10)$. With the previous example, we have $\lambda_n(137.1564) = \lambda_n(1371564) + 4\rho(10) = 19866327{-}0000$.

So, 23025842+0 actually represents $23025842 - \rho(10) = 0$. With this notation, Napier can add or subtract the zeros as if they were exponents, because he really adds or subtracts $\rho(10)$.

This cumbersome notation was certainly one of the main reasons which explained the move to decimal logarithms in which the characteristics become mere integers. The problem of Napier's logarithms (and also of the natural logarithms) is that their base is not equal to the base of the numeration.

6.5 Interpolation

In 1616, Briggs supplied a chapter in which he introduced interpolation methods.

6.6 Trigonometric computations

6.6.1 First example

The main purpose of Napier's table of logarithms was to simplify trigonometric calculations. In the *Descriptio*, Napier gives a number of rules for specific triangle problems. For instance, proposition 4 in chapter 2 of book 2 reads: *In any Triangle: the summe of the Logarithmes of any angle and side inclosing the same, is equall to the summe of the Logarithmes of the side, and the angle opposite to them.* [133, p. 35] When Napier writes "the logarithm of an angle," it implicitely means the "logarithm of the sine of the angle."

In other words, considering the figure 9 given by Napier, the proposition means for instance that

$$\lambda_n(\sin_n A) + \lambda_n(c) = \lambda_n(a) + \lambda_n(\sin_n C)$$

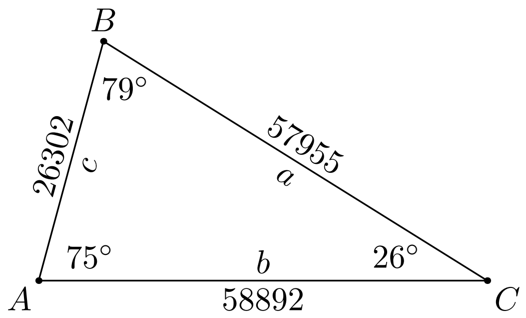


Figure 9: The application of logarithms to trigonometry.

This follows easily from the law of sines. We have

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Therefore

$$\ln(\sin A) = \ln(\sin C) + \ln a - \ln c$$

But $\lambda_n(x) = \lambda_n(1) - 10^7 \ln x$, hence

$$-10^7 \ln(\sin A) = -10^7 \ln(\sin C) - 10^7 \ln a + 10^7 \ln c$$

and

$$\lambda_n(\sin_n A) - \lambda_n(1) = (\lambda_n(\sin_n C) - \lambda_n(1)) + (\lambda_n(a) - \lambda_n(1)) - (\lambda_n(c) - \lambda_n(1))$$

which reduces to

$$\lambda_n(\sin_n A) = \lambda_n(\sin_n C) + \lambda_n(a) - \lambda_n(c)$$

or

$$\lambda_n(\sin_n A) + \lambda_n(c) = \lambda_n(a) + \lambda_n(\sin_n C)$$

which is Napier's result.

With the previous example, Napier (in 1614) obtains $\lambda_n(a) = 5454707{-}00$, $\lambda_n(\sin_n C) = 8246889$, $\lambda_n(c) = 13354921{-}00$, from which he computes $\lambda_n(\sin_n A) = 346684$, the resulting logarithm being pure, and this is nearly $\lambda_n(\sin_n 75°)$, and therefore $A = 75°$. Napier adds that the result would be $105°$ if the angle appeared to be obtuse.

If one checks Napier's table, only the value of $\lambda_n(\sin_n C) = 8246889$ can be found. For the two other values a and c, we find approaching values:

α	$\sin_n(\alpha)$	$\lambda_n(\sin_n(\alpha))$
$15°14'$	2627505	13365493
$15°15'$	2630312	13354817
$35°25'$	5795183	5455577
$35°26'$	5797553	5451488

$100a$ lies between $\sin_n(35°25')$ and $\sin_n(35°26')$, and $100c$ lies between $\sin_n(15°14')$ and $\sin_n(15°15')$.

It isn't clear how Napier's values were obtained, because his interpolation procedure gives different values:

$$10^7\frac{(2630312-2630200)}{2630312} < \lambda_n(2630200)-\lambda_n(2630312) < 10^7\frac{(2630312-2630200)}{2630200}$$

$$425.804\ldots < \lambda_n(2630200)-\lambda_n(2630312) < 425.823\ldots$$

$$\lambda_n(2630200) \approx \lambda_n(2630312)+425.8 = 13354817+425.8 = 13355243$$

which differs from Napier's 13354921.

$$10^7\frac{(5795500-5795183)}{5795500} < \lambda_n(5795183)-\lambda_n(5795500) < 10^7\frac{(5795500-5795183)}{5795183}$$

$$546.97\ldots < \lambda_n(5795183)-\lambda_n(5795500) < 547.00\ldots$$

$$\lambda_n(5795500) \approx \lambda_n(5795183)-547 = 5455577-547 = 5455030$$

which differs from Napier's 5454701.

Wright's example is exactly the same, but his logarithm values are all ten times smaller, which ensures that Napier's proposition is still valid.[29]

6.6.2 Second example

The third proposition of Book 2 in the *Descriptio* involves the *differential*, which corresponds to the logarithm of the tangent. The proposition reads thus [133, p. 33]: *In a right angled triangle the Logarithme of any legge is equall to the summe of the Differentiall of the opposite angle, and the Logarithme of the leg remaining.*

We can take as an example figure 10 which is Napier's example. We are given a triangle with three sides. Like before, if we set $\sigma_n(x) = \lambda_n(\sin_n x)$

[29]So, Wright has $\lambda_w(a) = 545471{-}0$, $\lambda_w(\sin_w C) = 824689$, $\lambda_w(c) = 1335492{-}0$, from which he deduces $\lambda_w(\sin_w A) = 34668$. Wright's translation mistakenly writes $34668{-}0$.

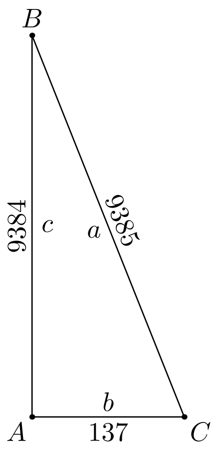


Figure 10: The computation of a right-angled triangle (not to scale).

and $\tau_n(x) = \textit{differential} = \sigma_n(x) - \sigma_n(90 - x)$, Napier's proposition then amounts to:

$$\begin{aligned}\lambda_n(b) &= \lambda_n(c) + \tau_n(B)\\ &= \lambda_n(c) + \sigma_n(B) - \sigma_n(90 - B)\\ &= \lambda_n(c) + \lambda_n(\sin_n B) - \lambda_n(\cos_n B)\end{aligned}$$

Since $\lambda_n(x) = 10^7(\ln(10^7) - \ln x) = \lambda_n(1) - 10^7 \ln x$, the above reduces to

$$\begin{aligned}\ln b &= \ln c + \ln(\sin B) - \ln(\cos B)\\ &= \ln c + \ln(\tan B)\end{aligned}$$

and therefore

$$b = c \tan B$$

which is correct.

Napier uses this proposition in order to find the angle B. Approximating 137000 by $\sin_n(47') = 136714$ and 938400 by $\sin_n(69°47') = 9383925$, he has (in 1614):

$$\begin{aligned}\tau_n(B) &= \lambda_n(b) - \lambda_n(c)\\ &= 42924534{-}000 - 635870{-}000\\ &= 42288664\end{aligned}$$

and since the *differential* is 42304768 for $0°50'$ and 42106711 for $0°51'$, Napier concludes that $B \approx 0°50'$.

Wright has the same triangle, with the same dimensions, but the values of the logarithms are $\lambda_w(b) = 4292453{-}00$ and $\lambda_w(c) = 63587{-}00$ and he finds $\tau_w(B) = 4228866$ and from this obtains $B = 0°50'11''$. It isn't clear how Wright obtained the latter value, which is incorrect. In Wright's table $\tau_w(0°50') = 4230477$ and $\tau_w(0°51') = 4210571$ and a linear interpolation gives $0°50'5''$ or $0°50'.08$.

Napier gave other examples, and also considered the use of logarithms for the resolution of spherical triangles. Napier devised a rule called of "circular parts," which was useful for such triangles [88, 105, 129].

7 Napier's scientific heritage

7.1 Napier's suggestions

Napier suggested several improvements to his tables. First, in the *Constructio* [137, p. 46], he suggested a finer grain construction of the three fundamental tables, which should ensure a greater accuracy. The new table 1 was to contain 100 steps, the new table 2 also 100 steps (compared to 50 in the initial scheme), and the new table 3 was to contain 100 columns of 35 steps (instead of 69 columns of 20 steps). The total number of intervals would therefore be $35 \cdot 10^6$ instead of $6.9 \cdot 10^6$.

Then, also in the *Constructio*, he suggested a better kind of logarithms, in which the logarithm of 1 is 0 and the logarithm of either 10 or $\frac{1}{10}$ is equal to 10. He then explained how these logarithms can be computed. He gave in particular an example whereby the decimal logarithm of 5 can be computed by repeated square root extractions and constructing a sequence approximating the sought result. Starting with $\log 10 = 1$ and $\log 1 = 0$, we compute $\log \sqrt{1 \times 10} = 0.5$, then $\log \sqrt{10 \times \sqrt{10}} = 0.75$, getting closer to $\log 5$ by using square roots of products of numbers greater and smaller than 5 [137, pp. 51 and 97–100], [35, pp. 736–737], [191, pp. 168–169].

7.2 Decimal logarithms

Napier's work was taken over and adapted by Briggs who published his first table as soon as 1617. The first to publish tables of decimal logarithms of trigonometric functions was Gunter in 1620 [76]. Briggs' main tables were published in 1624 and 1633, and Adriaan Vlacq published less accurate—but more extensive—tables in 1628 and 1633. These tables were then used as the basis of almost all later tables until the beginning of the 20th century.

7.3 Natural (Neperian) logarithms

These logarithms were introduced by Speidell in 1622 or 1623 [68, pp. 175–176] although a table that looked like one of natural logarithms already appeared in the appendix of the second edition of Wright's translation. This table was in fact a table extending Napier's short table for a number of simple ratios. The values in the table were the values of $\rho(1)$, $\rho(2)$, $\rho(3)$, etc., which are equal to $10^6 \ln(1)$, $10^6 \ln(2)$, $10^6 \ln(3)$, etc., but this table was never thought as being a table of logarithms. It is only a table for the differences of Napierian logarithms and it should not be historically interpreted otherwise, although Glaisher views them as the first publication of natural logarithms. A full analysis of this appendix which is thought to be by William Oughtred was given by Glaisher [68].

7.4 Other tables

1620 also saw the publication of Bürgi's tables, albeit without a description of their use.[30] Kepler was familiar both with Napier's logarithms and with Bürgi's work, and he also published tables of his own.

At about the same time, Benjamin Ursinus and John Speidell also published tables extending Napier's tables, see in particular Cantor [35, p. 739–743]. Speidell's table of logarithms was reproduced by Maseres [111].

7.5 Slide rules

A very important consequence of Napier's invention was the development of the slide rule. First came Gunter's scale. Figure 11 shows a logarithmic scale, which was one of the scales found on the scale named after Edmund Gunter (1581–1626) who invented it in 1620. Multiplications or divisions could be done with this rule using a pair of dividers. Around 1622, William Oughtred (1574–1660) had the idea of using two logarithmic scales and putting them side by side (figure 12), which made it possible to dispense with the dividers. Oughtred's initial design used circular scales [145, 201], and pairs of straight scales were only introduced later.

[30] For an analysis of Bürgi's tables, see our reconstruction [167]. Although Bürgi's tables can be viewed as tables of logarithms, Bürgi did not reach the abstract notion developed by Napier, and should not be considered as a co-inventor of logarithms.

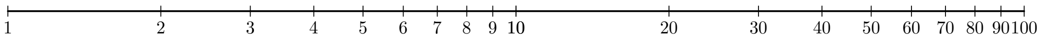


Figure 11: A logarithmic scale from 1 to 100.

It is easy to see how these scales are used and why they work. In figure 11, the scale goes from 1 to 100, and the positions of the numbers are proportional to their logarithm. In other words, the distance between n and 1 is proportional to $\log(n)$. The distance between 1 and 10 is the same as between 10 and 100, because their logarithms are equidifferent. A given divider opening corresponds to every ratio. Multiplying by 2 corresponds to the distance between 1 and 2, which is also the distance between 2 and 4, between 4 and 8, between 10 and 20, etc. A pair of divider can therefore easily be used to perform multiplications or divisions.

In figure 12, two such scales are put in parallel and the 1 of the first rule is put above some position of the second scale, here $a = 2.5$. Since a given ratio corresponds to the same linear distance on both scales, the value c facing a certain value b of the first scale, for instance 6, is such that $\frac{c}{a} = \frac{b}{1}$, and therefore $c = ab$. The same arrangement can be used for divisions, and dividers are no longer needed.

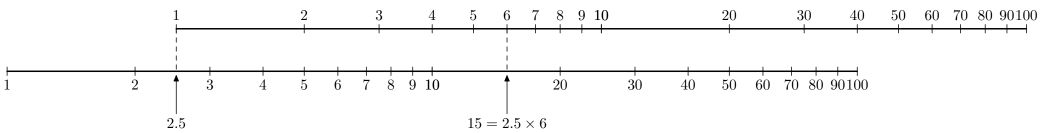


Figure 12: Two logarithmic scales showing the computation of $2.5 \times 6 = 15$.

On the history of the slide rule, we refer the reader to Cajori's articles and books [30, 31, 34][31] or Stoll's popular account in the *Scientific American* [186].

On the construction of the logarithmic lines on such a scale, see also Robertson [155] and Nicholson [143].

A good source for more recent information on the history of slide rules is the *Journal of the Oughtred Society*.

It is important to remember that many "popular" encyclopædias covering a large subject are bound to contain many errors, even when written by reputed mathematicians. Two interesting articles worth reading in this context are those of Mautz [114] and Miller [124].

[31] In his 1909 book, Cajori first attributes the invention of the slide rule to Wingate, but then corrects himself in an addenda.

8 Note on the recalculation of the tables

8.1 Auxiliary tables

In the tables closing this document, we have computed the a_i, b_i and $c_{i,j}$ exactly and the logarithms using Napier's averaging method. Hence, we took $l_1(a_1) = 1.00000005$, $l_2(b_2) = 100.0005$, $l_3(c_{1,0}) = 5001.2504168$ and $l_3(c_{0,1}) = 100503.358535$.

8.2 Main tables

There are three volumes accompanying this study. The first[32] gives the ideal recomputation of Napier's logarithms, that is the values of $\sin_n \alpha$ and $\lambda_n(\alpha)$ with an exact computation.

The other two volumes[33] provide approximations of Napier's tables, assuming that Napier did not make any mistake in the auxiliary tables and assuming he did not round any value. These two volumes have been constructed using Napier's radix table, as well as his method of interpolation. Because of the latter, the values in these two tables do differ from the ideal table.

Differences between Napier's actual table and our tables are due to errors on sines, to rounding, to errors in the auxiliary tables, or to other errors during the interpolation. It seemed pointless to try to mimick all these errors.

It should also be noted that we did not use the trigonometric relations to compute the logarithms for angles under 45°, because Napier would probably not have used them, if he had been aware of the propagation of errors, or if he had had the time to do more exact computations. It would have seemed strange to correct the errors in Napier's auxiliary tables, and at the same time to use an error-prone trigonometric computation, since the combined result of this procedure would still have been far from Napier's actual table.

We have *not* rounded the sines before taking the logarithms, as it seems that Napier used more accurate values of the sines for the small angles than those which are given in his table. He may have used the rounded values for larger angles.

We might try to produce a more faithful approximation of Napier's table in the future. Such an approximation would in particular take rounding into account in the computation of the sequences a_i, b_i, etc.

[32] Volume `napier1614idealdoc`.

[33] Volumes `napier1614doc` and `napier1616doc`.

The reconstruction of the 1616 table was obtained by rounding the reconstruction of the 1614 table. Wright didn't recompute any value of the table.

9 Acknowledgements

It is a pleasure to thank Ian Bruce who provided very useful translations of Napier's *Descriptio* and *Constructio* and helped me to clarify some details about Napier's work.

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The following list covers the most important references[34] related to Napier's tables. Not all items of this list are mentioned in the text, and the sources which have not been seen are marked so. We have added notes about the contents of the articles in certain cases.

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[34]**Note on the titles of the works:** Original titles come with many idiosyncrasies and features (line splitting, size, fonts, etc.) which can often not be reproduced in a list of references. It has therefore seemed pointless to capitalize works according to conventions which not only have no relation with the original work, but also do not restore the title entirely. In the following list of references, most title words (except in German) will therefore be left uncapitalized. The names of the authors have also been homogenized and initials expanded, as much as possible.

The reader should keep in mind that this list is not meant as a facsimile of the original works. The original style information could no doubt have been added as a note, but we have not done it here.

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10 Napier's construction tables

10.1 First table

i	Number a_i	Logarithm $l_1(a_i)$	i	Number a_i	Logarithm $l_1(a_i)$
0	10000000.0000000	0.0000000	25	9999975.0000300	25.0000013
1	9999999.0000000	1.0000001	26	9999974.0000325	26.0000013
2	9999998.0000001	2.0000001	27	9999973.0000351	27.0000014
3	9999997.0000003	3.0000002	28	9999972.0000378	28.0000014
4	9999996.0000006	4.0000002	29	9999971.0000406	29.0000015
5	9999995.0000010	5.0000003	30	9999970.0000435	30.0000015
6	9999994.0000015	6.0000003	31	9999969.0000465	31.0000016
7	9999993.0000021	7.0000004	32	9999968.0000496	32.0000016
8	9999992.0000028	8.0000004	33	9999967.0000528	33.0000017
9	9999991.0000036	9.0000005	34	9999966.0000561	34.0000017
10	9999990.0000045	10.0000005	35	9999965.0000595	35.0000018
11	9999989.0000055	11.0000006	36	9999964.0000630	36.0000018
12	9999988.0000066	12.0000006	37	9999963.0000666	37.0000019
13	9999987.0000078	13.0000007	38	9999962.0000703	38.0000019
14	9999986.0000091	14.0000007	39	9999961.0000741	39.0000020
15	9999985.0000105	15.0000008	40	9999960.0000780	40.0000020
16	9999984.0000120	16.0000008	41	9999959.0000820	41.0000021
17	9999983.0000136	17.0000009	42	9999958.0000861	42.0000021
18	9999982.0000153	18.0000009	43	9999957.0000903	43.0000022
19	9999981.0000171	19.0000010	44	9999956.0000946	44.0000022
20	9999980.0000190	20.0000010	45	9999955.0000990	45.0000023
21	9999979.0000210	21.0000011	46	9999954.0001035	46.0000023
22	9999978.0000231	22.0000011	47	9999953.0001081	47.0000024
23	9999977.0000253	23.0000012	48	9999952.0001128	48.0000024
24	9999976.0000276	24.0000012	49	9999951.0001176	49.0000025
25	9999975.0000300	25.0000013	50	9999950.0001225	50.0000025

i	Number a_i	Logarithm $l_1(a_i)$	i	Number a_i	Logarithm $l_1(a_i)$
50	9999950.0001225	50.0000025	75	9999925.0002775	75.0000038
51	9999949.0001275	51.0000026	76	9999924.0002850	76.0000038
52	9999948.0001326	52.0000026	77	9999923.0002926	77.0000039
53	9999947.0001378	53.0000027	78	9999922.0003003	78.0000039
54	9999946.0001431	54.0000027	79	9999921.0003081	79.0000040
55	9999945.0001485	55.0000028	80	9999920.0003160	80.0000040
56	9999944.0001540	56.0000028	81	9999919.0003240	81.0000041
57	9999943.0001596	57.0000029	82	9999918.0003321	82.0000041
58	9999942.0001653	58.0000029	83	9999917.0003403	83.0000042
59	9999941.0001711	59.0000030	84	9999916.0003486	84.0000042
60	9999940.0001770	60.0000030	85	9999915.0003570	85.0000043
61	9999939.0001830	61.0000031	86	9999914.0003655	86.0000043
62	9999938.0001891	62.0000031	87	9999913.0003741	87.0000044
63	9999937.0001953	63.0000032	88	9999912.0003828	88.0000044
64	9999936.0002016	64.0000032	89	9999911.0003916	89.0000045
65	9999935.0002080	65.0000033	90	9999910.0004005	90.0000045
66	9999934.0002145	66.0000033	91	9999909.0004095	91.0000046
67	9999933.0002211	67.0000034	92	9999908.0004186	92.0000046
68	9999932.0002278	68.0000034	93	9999907.0004278	93.0000047
69	9999931.0002346	69.0000035	94	9999906.0004371	94.0000047
70	9999930.0002415	70.0000035	95	9999905.0004465	95.0000048
71	9999929.0002485	71.0000036	96	9999904.0004560	96.0000048
72	9999928.0002556	72.0000036	97	9999903.0004656	97.0000049
73	9999927.0002628	73.0000037	98	9999902.0004753	98.0000049
74	9999926.0002701	74.0000037	99	9999901.0004851	99.0000050
75	9999925.0002775	75.0000038	100	9999900.0004950	100.0000050

10.2 Second table

i	Number b_i	Logarithm $l_2(b_i)$	i	Number b_i	Logarithm $l_2(b_i)$
0	10000000.000000	0.0000000	25	9997500.299977	2500.0125000
1	9999900.000000	100.0005000	26	9997400.324974	2600.0130000
2	9999800.001000	200.0010000	27	9997300.350971	2700.0135000
3	9999700.003000	300.0015000	28	9997200.377967	2800.0140000
4	9999600.006000	400.0020000	29	9997100.405963	2900.0145000
5	9999500.010000	500.0025000	30	9997000.434959	3000.0150000
6	9999400.015000	600.0030000	31	9996900.464955	3100.0155000
7	9999300.021000	700.0035000	32	9996800.495950	3200.0160000
8	9999200.027999	800.0040000	33	9996700.527945	3300.0165000
9	9999100.035999	900.0045000	34	9996600.560940	3400.0170000
10	9999000.044999	1000.0050000	35	9996500.594935	3500.0175000
11	9998900.054998	1100.0055000	36	9996400.629929	3600.0180000
12	9998800.065998	1200.0060000	37	9996300.665922	3700.0185000
13	9998700.077997	1300.0065000	38	9996200.702916	3800.0190000
14	9998600.090996	1400.0070000	39	9996100.740909	3900.0195000
15	9998500.104995	1500.0075000	40	9996000.779901	4000.0200000
16	9998400.119994	1600.0080000	41	9995900.819893	4100.0205000
17	9998300.135993	1700.0085000	42	9995800.860885	4200.0210000
18	9998200.152992	1800.0090000	43	9995700.902877	4300.0215000
19	9998100.170990	1900.0095000	44	9995600.945868	4400.0220000
20	9998000.189989	2000.0100000	45	9995500.989858	4500.0225000
21	9997900.209987	2100.0105000	46	9995401.034848	4600.0230000
22	9997800.230985	2200.0110000	47	9995301.080838	4700.0235000
23	9997700.252982	2300.0115000	48	9995201.127827	4800.0240000
24	9997600.275980	2400.0120000	49	9995101.175816	4900.0245000
25	9997500.299977	2500.0125000	50	9995001.224804	5000.0250000

10.3 Third table

Column 0			Column 1			Column 2		
i	Number $c_{i,0}$	Logarithm $l_3(c_{i,0})$	i	Number $c_{i,1}$	Logarithm $l_3(c_{i,1})$	i	Number $c_{i,2}$	Logarithm $l_3(c_{i,2})$
0	10000000.000000	0.0	0	9900000.000000	100503.4	0	9801000.000000	201006.7
1	9995000.000000	5001.3	1	9895050.000000	105504.6	1	9796099.500000	206008.0
2	9990002.500000	10002.5	2	9890102.475000	110505.9	2	9791201.450250	211009.2
3	9985007.498750	15003.8	3	9885157.423762	115507.1	3	9786305.849525	216010.5
4	9980014.995001	20005.0	4	9880214.845051	120508.4	4	9781412.696600	221011.7
5	9975024.987503	25006.3	5	9875274.737628	125509.6	5	9776521.990252	226013.0
6	9970037.475009	30007.5	6	9870337.100259	130510.9	6	9771633.729257	231014.2
7	9965052.456272	35008.8	7	9865401.931709	135512.1	7	9766747.912392	236015.5
8	9960069.930044	40010.0	8	9860469.230743	140513.4	8	9761864.538436	241016.7
9	9955089.895079	45011.3	9	9855538.996128	145514.6	9	9756983.606167	246018.0
10	9950112.350131	50012.5	10	9850611.226630	150515.9	10	9752105.114364	251019.2
11	9945137.293956	55013.8	11	9845685.921017	155517.1	11	9747229.061806	256020.5
12	9940164.725309	60015.0	12	9840763.078056	160518.4	12	9742355.447275	261021.7
13	9935194.642946	65016.3	13	9835842.696517	165519.6	13	9737484.269552	266023.0
14	9930227.045625	70017.5	14	9830924.775169	170520.9	14	9732615.527417	271024.2
15	9925261.932102	75018.8	15	9826009.312781	175522.1	15	9727749.219653	276025.5
16	9920299.301136	80020.0	16	9821096.308125	180523.4	16	9722885.345044	281026.7
17	9915339.151486	85021.3	17	9816185.759971	185524.6	17	9718023.902371	286028.0
18	9910381.481910	90022.5	18	9811277.667091	190525.9	18	9713164.890420	291029.2
19	9905426.291169	95023.8	19	9806372.028257	195527.1	19	9708308.307975	296030.5
20	9900473.578023	100025.0	20	9801468.842243	200528.4	20	9703454.153821	301031.7

Column 3			Column 4			Column 5		
i	Number $c_{i,3}$	Logarithm $l_3(c_{i,3})$	i	Number $c_{i,4}$	Logarithm $l_3(c_{i,4})$	i	Number $c_{i,5}$	Logarithm $l_3(c_{i,5})$
0	9702990.000000	301510.1	0	9605960.100000	402013.4	0	9509900.499000	502516.8
1	9698138.505000	306511.3	1	9601157.119950	407014.7	1	9505145.548750	507518.0
2	9693289.435747	311512.6	2	9596356.541390	412015.9	2	9500392.975976	512519.3
3	9688442.791030	316513.8	3	9591558.363119	417017.2	3	9495642.779488	517520.5
4	9683598.569634	321515.1	4	9586762.583938	422018.4	4	9490894.958098	522521.8
5	9678756.770349	326516.3	5	9581969.202646	427019.7	5	9486149.510619	527523.0
6	9673917.391964	331517.6	6	9577178.218044	432020.9	6	9481406.435864	532524.3
7	9669080.433268	336518.8	7	9572389.628935	437022.2	7	9476665.732646	537525.5
8	9664245.893052	341520.1	8	9567603.434121	442023.4	8	9471927.399780	542526.8
9	9659413.770105	346521.3	9	9562819.632404	447024.7	9	9467191.436080	547528.0
10	9654584.063220	351522.6	10	9558038.222588	452025.9	10	9462457.840362	552529.3
11	9649756.771188	356523.8	11	9553259.203476	457027.2	11	9457726.611442	557530.5
12	9644931.892803	361525.1	12	9548482.573875	462028.4	12	9452997.748136	562531.8
13	9640109.426856	366526.3	13	9543708.332588	467029.7	13	9448271.249262	567533.0
14	9635289.372143	371527.6	14	9538936.478421	472030.9	14	9443547.113637	572534.3
15	9630471.727457	376528.8	15	9534167.010182	477032.2	15	9438825.340080	577535.5
16	9625656.491593	381530.1	16	9529399.926677	482033.4	16	9434105.927410	582536.8
17	9620843.663347	386531.3	17	9524635.226714	487034.7	17	9429388.874447	587538.0
18	9616033.241516	391532.6	18	9519872.909100	492035.9	18	9424674.180009	592539.3
19	9611225.224895	396533.8	19	9515112.972646	497037.2	19	9419961.842919	597540.6
20	9606419.612282	401535.1	20	9510355.416160	502038.4	20	9415251.861998	602541.8

Column 6			Column 7			Column 8		
	Number	Logarithm		Number	Logarithm		Number	Logarithm
i	$c_{i,6}$	$l_3(c_{i,6})$	i	$c_{i,7}$	$l_3(c_{i,7})$	i	$c_{i,8}$	$l_3(c_{i,8})$
0	9414801.494010	603020.2	0	9320653.479070	703523.5	0	9227446.944279	804026.9
1	9410094.093263	608021.4	1	9315993.152330	708524.8	1	9222833.220807	809028.1
2	9405389.046216	613022.7	2	9311335.155754	713526.0	2	9218221.804197	814029.4
3	9400686.351693	618023.9	3	9306679.488176	718527.3	3	9213612.693295	819030.6
4	9395986.008517	623025.2	4	9302026.148432	723528.5	4	9209005.886948	824031.9
5	9391288.015513	628026.4	5	9297375.135358	728529.8	5	9204401.384004	829033.1
6	9386592.371505	633027.7	6	9292726.447790	733531.0	6	9199799.183312	834034.4
7	9381899.075320	638028.9	7	9288080.084566	738532.3	7	9195199.283721	839035.6
8	9377208.125782	643030.2	8	9283436.044524	743533.5	8	9190601.684079	844036.9
9	9372519.521719	648031.4	9	9278794.326502	748534.8	9	9186006.383237	849038.1
10	9367833.261958	653032.7	10	9274154.929339	753536.0	10	9181413.380045	854039.4
11	9363149.345327	658033.9	11	9269517.851874	758537.3	11	9176822.673355	859040.6
12	9358467.770655	663035.2	12	9264883.092948	763538.5	12	9172234.262019	864041.9
13	9353788.536769	668036.4	13	9260250.651402	768539.8	13	9167648.144888	869043.1
14	9349111.642501	673037.7	14	9255620.526076	773541.0	14	9163064.320815	874044.4
15	9344437.086680	678038.9	15	9250992.715813	778542.3	15	9158482.788655	879045.6
16	9339764.868136	683040.2	16	9246367.219455	783543.5	16	9153903.547260	884046.9
17	9335094.985702	688041.4	17	9241744.035845	788544.8	17	9149326.595487	889048.1
18	9330427.438209	693042.7	18	9237123.163827	793546.0	18	9144751.932189	894049.4
19	9325762.224490	698043.9	19	9232504.602245	798547.3	19	9140179.556223	899050.6
20	9321099.343378	703045.2	20	9227888.349944	803548.5	20	9135609.466445	904051.9

Column 9			Column 10			Column 11		
	Number	Logarithm		Number	Logarithm		Number	Logarithm
i	$c_{i,9}$	$l_3(c_{i,9})$	i	$c_{i,10}$	$l_3(c_{i,10})$	i	$c_{i,11}$	$l_3(c_{i,11})$
0	9135172.474836	904530.2	0	9043820.750088	1005033.6	0	8953382.542587	1105536.9
1	9130604.888599	909531.5	1	9039298.839713	1010034.8	1	8948905.851316	1110538.2
2	9126039.586155	914532.7	2	9034779.190293	1015036.1	2	8944431.398390	1115539.4
3	9121476.566362	919534.0	3	9030261.800698	1020037.3	3	8939959.182691	1120540.7
4	9116915.828078	924535.2	4	9025746.669798	1025038.6	4	8935489.203100	1125541.9
5	9112357.370164	929536.5	5	9021233.796463	1030039.8	5	8931021.458498	1130543.2
6	9107801.191479	934537.7	6	9016723.179565	1035041.1	6	8926555.947769	1135544.4
7	9103247.290884	939539.0	7	9012214.817975	1040042.3	7	8922092.669795	1140545.7
8	9098695.667238	944540.2	8	9007708.710566	1045043.6	8	8917631.623460	1145546.9
9	9094146.319405	949541.5	9	9003204.856210	1050044.8	9	8913172.807648	1150548.2
10	9089599.246245	954542.7	10	8998703.253782	1055046.1	10	8908716.221245	1155549.4
11	9085054.446622	959544.0	11	8994203.902155	1060047.3	11	8904261.863134	1160550.7
12	9080511.919398	964545.2	12	8989706.800204	1065048.6	12	8899809.732202	1165551.9
13	9075971.663439	969546.5	13	8985211.946804	1070049.8	13	8895359.827336	1170553.2
14	9071433.677607	974547.7	14	8980719.340831	1075051.1	14	8890912.147423	1175554.4
15	9066897.960768	979549.0	15	8976228.981160	1080052.3	15	8886466.691349	1180555.7
16	9062364.511788	984550.2	16	8971740.866670	1085053.6	16	8882023.458003	1185557.0
17	9057833.329532	989551.5	17	8967254.996237	1090054.8	17	8877582.446274	1190558.2
18	9053304.412867	994552.7	18	8962771.368738	1095056.1	18	8873143.655051	1195559.5
19	9048777.760661	999554.0	19	8958289.983054	1100057.3	19	8868707.083224	1200560.7
20	9044253.371780	1004555.2	20	8953810.838063	1105058.6	20	8864272.729682	1205562.0

Column 12			Column 13			Column 14		
	Number	Logarithm		Number	Logarithm		Number	Logarithm
i	$c_{i,12}$	$l_3(c_{i,12})$	i	$c_{i,13}$	$l_3(c_{i,13})$	i	$c_{i,14}$	$l_3(c_{i,14})$
0	8863848.717161	1206040.3	0	8775210.229990	1306543.7	0	8687458.127690	1407047.0
1	8859416.792803	1211041.6	1	8770822.624875	1311544.9	1	8683114.398626	1412048.3
2	8854987.084406	1216042.8	2	8766437.213562	1316546.2	2	8678772.841427	1417049.5
3	8850559.590864	1221044.1	3	8762053.994955	1321547.4	3	8674433.455006	1422050.8
4	8846134.311069	1226045.3	4	8757672.967958	1326548.7	4	8670096.238278	1427052.0
5	8841711.243913	1231046.6	5	8753294.131474	1331549.9	5	8665761.190159	1432053.3
6	8837290.388291	1236047.8	6	8748917.484408	1336551.2	6	8661428.309564	1437054.5
7	8832871.743097	1241049.1	7	8744543.025666	1341552.4	7	8657097.595409	1442055.8
8	8828455.307225	1246050.3	8	8740170.754153	1346553.7	8	8652769.046612	1447057.0
9	8824041.079572	1251051.6	9	8735800.668776	1351554.9	9	8648442.662088	1452058.3
10	8819629.059032	1256052.8	10	8731432.768442	1356556.2	10	8644118.440757	1457059.5
11	8815219.244503	1261054.1	11	8727067.052058	1361557.4	11	8639796.381537	1462060.8
12	8810811.634880	1266055.3	12	8722703.518532	1366558.7	12	8635476.483346	1467062.0
13	8806406.229063	1271056.6	13	8718342.166772	1371559.9	13	8631158.745105	1472063.3
14	8802003.025948	1276057.8	14	8713982.995689	1376561.2	14	8626843.165732	1477064.5
15	8797602.024435	1281059.1	15	8709626.004191	1381562.4	15	8622529.744149	1482065.8
16	8793203.223423	1286060.3	16	8705271.191189	1386563.7	16	8618218.479277	1487067.0
17	8788806.621811	1291061.6	17	8700918.555593	1391564.9	17	8613909.370037	1492068.3
18	8784412.218501	1296062.8	18	8696568.096316	1396566.2	18	8609602.415352	1497069.5
19	8780020.012391	1301064.1	19	8692219.812267	1401567.4	19	8605297.614145	1502070.8
20	8775630.002385	1306065.3	20	8687873.702361	1406568.7	20	8600994.965338	1507072.0

Column 15			Column 16			Column 17		
i	Number $c_{i,15}$	Logarithm $l_3(c_{i,15})$	i	Number $c_{i,16}$	Logarithm $l_3(c_{i,16})$	i	Number $c_{i,17}$	Logarithm $l_3(c_{i,17})$
0	8600583.546413	1507550.4	0	8514577.710949	1608053.7	0	8429431.933839	1708557.1
1	8596283.254640	1512551.6	1	8510320.422093	1613055.0	1	8425217.217872	1713558.3
2	8591985.113012	1517552.9	2	8506065.261882	1618056.2	2	8421004.609263	1718559.6
3	8587689.120456	1522554.1	3	8501812.229251	1623057.5	3	8416794.106959	1723560.8
4	8583395.275896	1527555.4	4	8497561.323137	1628058.7	4	8412585.709905	1728562.1
5	8579103.578258	1532556.6	5	8493312.542475	1633060.0	5	8408379.417050	1733563.3
6	8574814.026469	1537557.9	6	8489065.886204	1638061.2	6	8404175.227342	1738564.6
7	8570526.619455	1542559.1	7	8484821.353261	1643062.5	7	8399973.139728	1743565.8
8	8566241.356146	1547560.4	8	8480578.942584	1648063.7	8	8395773.153158	1748567.1
9	8561958.235468	1552561.6	9	8476338.653113	1653065.0	9	8391575.266582	1753568.3
10	8557677.256350	1557562.9	10	8472100.483786	1658066.2	10	8387379.478948	1758569.6
11	8553398.417722	1562564.1	11	8467864.433544	1663067.5	11	8383185.789209	1763570.8
12	8549121.718513	1567565.4	12	8463630.501328	1668068.7	12	8378994.196314	1768572.1
13	8544847.157653	1572566.6	13	8459398.686077	1673070.0	13	8374804.699216	1773573.4
14	8540574.734075	1577567.9	14	8455168.986734	1678071.2	14	8370617.296867	1778574.6
15	8536304.446708	1582569.1	15	8450941.402241	1683072.5	15	8366431.988218	1783575.9
16	8532036.294484	1587570.4	16	8446715.931539	1688073.7	16	8362248.772224	1788577.1
17	8527770.276337	1592571.6	17	8442492.573574	1693075.0	17	8358067.647838	1793578.4
18	8523506.391199	1597572.9	18	8438271.327287	1698076.2	18	8353888.614014	1798579.6
19	8519244.638003	1602574.1	19	8434052.191623	1703077.5	19	8349711.669707	1803580.9
20	8514985.015684	1607575.4	20	8429835.165527	1708078.7	20	8345536.813872	1808582.1

Column 18			Column 19			Column 20		
i	Number $c_{i,18}$	Logarithm $l_3(c_{i,18})$	i	Number $c_{i,19}$	Logarithm $l_3(c_{i,19})$	i	Number $c_{i,20}$	Logarithm $l_3(c_{i,20})$
0	8345137.614501	1809060.5	0	8261686.238356	1909563.8	0	8179069.375972	2010067.2
1	8340965.045694	1814061.7	1	8257555.395237	1914565.1	1	8174979.841284	2015068.4
2	8336794.563171	1819063.0	2	8253426.617539	1919566.3	2	8170892.351364	2020069.7
3	8332626.165889	1824064.2	3	8249299.904230	1924567.6	3	8166806.905188	2025070.9
4	8328459.852806	1829065.5	4	8245175.254278	1929568.8	4	8162723.501735	2030072.2
5	8324295.622880	1834066.7	5	8241052.666651	1934570.1	5	8158642.139985	2035073.4
6	8320133.475068	1839068.0	6	8236932.140318	1939571.3	6	8154562.818915	2040074.7
7	8315973.408331	1844069.2	7	8232813.674248	1944572.6	7	8150485.537505	2045075.9
8	8311815.421627	1849070.5	8	8228697.267410	1949573.8	8	8146410.294736	2050077.2
9	8307659.513916	1854071.7	9	8224582.918777	1954575.1	9	8142337.089589	2055078.4
10	8303505.684159	1859073.0	10	8220470.627317	1959576.3	10	8138265.921044	2060079.7
11	8299353.931317	1864074.2	11	8216360.392004	1964577.6	11	8134196.788084	2065080.9
12	8295204.254351	1869075.5	12	8212252.211808	1969578.8	12	8130129.689690	2070082.2
13	8291056.652224	1874076.7	13	8208146.085702	1974580.1	13	8126064.624845	2075083.4
14	8286911.123898	1879078.0	14	8204042.012659	1979581.3	14	8122001.592532	2080084.7
15	8282767.668336	1884079.2	15	8199939.991653	1984582.6	15	8117940.591736	2085085.9
16	8278626.284502	1889080.5	16	8195840.021657	1989583.8	16	8113881.621440	2090087.2
17	8274486.971360	1894081.7	17	8191742.101646	1994585.1	17	8109824.680629	2095088.4
18	8270349.727874	1899083.0	18	8187646.230595	1999586.3	18	8105769.768289	2100089.7
19	8266214.553010	1904084.2	19	8183552.407480	2004587.6	19	8101716.883405	2105090.9
20	8262081.445733	1909085.5	20	8179460.631276	2009588.8	20	8097666.024963	2110092.2

Column 21			Column 22			Column 23		
i	Number $c_{i,21}$	Logarithm $l_3(c_{i,21})$	i	Number $c_{i,22}$	Logarithm $l_3(c_{i,22})$	i	Number $c_{i,23}$	Logarithm $l_3(c_{i,23})$
0	8097278.682213	2110570.5	0	8016305.895390	2211073.9	0	7936142.836437	2311577.2
1	8093230.042871	2115571.8	1	8012297.742443	2216075.1	1	7932174.765018	2316578.5
2	8089183.427850	2120573.0	2	8008291.593572	2221076.4	2	7928208.677636	2321579.7
3	8085138.836136	2125574.3	3	8004287.447775	2226077.6	3	7924244.573297	2326581.0
4	8081096.266718	2130575.5	4	8000285.304051	2231078.9	4	7920282.451010	2331582.2
5	8077055.718585	2135576.8	5	7996285.161399	2236080.1	5	7916322.309785	2336583.5
6	8073017.190725	2140578.0	6	7992287.018818	2241081.4	6	7912364.148630	2341584.7
7	8068980.682130	2145579.3	7	7988290.875309	2246082.6	7	7908407.966556	2346586.0
8	8064946.191789	2150580.5	8	7984296.729871	2251083.9	8	7904453.762572	2351587.2
9	8060913.718693	2155581.8	9	7980304.581506	2256085.1	9	7900501.535691	2356588.5
10	8056883.261834	2160583.0	10	7976314.429215	2261086.4	10	7896551.284923	2361589.8
11	8052854.820203	2165584.3	11	7972326.272001	2266087.6	11	7892603.009281	2366591.0
12	8048828.392793	2170585.5	12	7968340.108865	2271088.9	12	7888656.707776	2371592.3
13	8044803.978596	2175586.8	13	7964355.938810	2276090.1	13	7884712.379422	2376593.5
14	8040781.576607	2180588.0	14	7960373.760841	2281091.4	14	7880770.023233	2381594.8
15	8036761.185819	2185589.3	15	7956393.573961	2286092.6	15	7876829.638221	2386596.0
16	8032742.805226	2190590.5	16	7952415.377174	2291093.9	16	7872891.223402	2391597.3
17	8028726.433823	2195591.8	17	7948439.169485	2296095.1	17	7868954.777790	2396598.5
18	8024712.070606	2200593.0	18	7944464.949900	2301096.4	18	7865020.300401	2401599.8
19	8020699.714571	2205594.3	19	7940492.717425	2306097.6	19	7861087.790251	2406601.0
20	8016689.364714	2210595.5	20	7936522.471067	2311098.9	20	7857157.246356	2411602.3

Column 24			Column 25			Column 26		
	Number	Logarithm		Number	Logarithm		Number	Logarithm
i	$c_{i,24}$	$l_3(c_{i,24})$	i	$c_{i,25}$	$l_3(c_{i,25})$	i	$c_{i,26}$	$l_3(c_{i,26})$
0	7856781.408072	2412080.6	0	7778213.593991	2512584.0	0	7700431.458052	2613087.3
1	7852853.017368	2417081.9	1	7774324.487194	2517585.2	1	7696581.242323	2618088.6
2	7848926.590859	2422083.1	2	7770437.324951	2522586.5	2	7692732.951701	2623089.8
3	7845002.127564	2427084.4	3	7766552.106288	2527587.7	3	7688886.585226	2628091.1
4	7841079.626500	2432085.6	4	7762668.830235	2532589.0	4	7685042.141933	2633092.3
5	7837159.086687	2437086.9	5	7758787.495820	2537590.2	5	7681199.620862	2638093.6
6	7833240.507144	2442088.1	6	7754908.102072	2542591.5	6	7677359.021052	2643094.8
7	7829323.886890	2447089.4	7	7751030.648021	2547592.7	7	7673520.341541	2648096.1
8	7825409.224947	2452090.6	8	7747155.132697	2552594.0	8	7669683.581370	2653097.3
9	7821496.520334	2457091.9	9	7743281.555131	2557595.2	9	7665848.739580	2658098.6
10	7817585.772074	2462093.1	10	7739409.914353	2562596.5	10	7662015.815210	2663099.8
11	7813676.979188	2467094.4	11	7735540.209396	2567597.7	11	7658184.807302	2668101.1
12	7809770.140698	2472095.6	12	7731672.439291	2572599.0	12	7654355.714898	2673102.3
13	7805865.255628	2477096.9	13	7727806.603072	2577600.2	13	7650528.537041	2678103.6
14	7801962.323000	2482098.1	14	7723942.699770	2582601.5	14	7646703.272773	2683104.8
15	7798061.341839	2487099.4	15	7720080.728420	2587602.7	15	7642879.921136	2688106.1
16	7794162.311168	2492100.6	16	7716220.688056	2592604.0	16	7639058.481176	2693107.3
17	7790265.230012	2497101.9	17	7712362.577712	2597605.2	17	7635238.951935	2698108.6
18	7786370.097397	2502103.1	18	7708506.396423	2602606.5	18	7631421.332459	2703109.8
19	7782476.912349	2507104.4	19	7704652.143225	2607607.7	19	7627605.621793	2708111.1
20	7778585.673892	2512105.6	20	7700799.817153	2612609.0	20	7623791.818982	2713112.3

Column 27			Column 28			Column 29		
	Number	Logarithm		Number	Logarithm		Number	Logarithm
i	$c_{i,27}$	$l_3(c_{i,27})$	i	$c_{i,28}$	$l_3(c_{i,28})$	i	$c_{i,29}$	$l_3(c_{i,29})$
0	7623427.143471	2713590.7	0	7547192.872036	2814094.0	0	7471720.943316	2914597.4
1	7619615.429899	2718591.9	1	7543419.275600	2819095.3	1	7467985.082844	2919598.6
2	7615805.622184	2723593.2	2	7539647.565963	2824096.5	2	7464251.090303	2924599.9
3	7611997.719373	2728594.4	3	7535877.742180	2829097.8	3	7460518.964758	2929601.1
4	7608191.720514	2733595.7	4	7532109.803308	2834099.0	4	7456788.705275	2934602.4
5	7604387.624653	2738596.9	5	7528343.748407	2839100.3	5	7453060.310923	2939603.6
6	7600585.430841	2743598.2	6	7524579.576533	2844101.5	6	7449333.780767	2944604.9
7	7596785.138126	2748599.4	7	7520817.286744	2849102.8	7	7445609.113877	2949606.2
8	7592986.745557	2753600.7	8	7517056.878101	2854104.0	8	7441886.309320	2954607.4
9	7589190.252184	2758601.9	9	7513298.349662	2859105.3	9	7438165.366165	2959608.7
10	7585395.657058	2763603.2	10	7509541.700487	2864106.5	10	7434446.283482	2964609.9
11	7581602.959229	2768604.4	11	7505786.929637	2869107.8	11	7430729.060340	2969611.2
12	7577812.157749	2773605.7	12	7502034.036172	2874109.0	12	7427013.695810	2974612.4
13	7574023.251671	2778606.9	13	7498283.019154	2879110.3	13	7423300.188962	2979613.7
14	7570236.240045	2783608.2	14	7494533.877644	2884111.5	14	7419588.538868	2984614.9
15	7566451.121925	2788609.4	15	7490786.610706	2889112.8	15	7415878.744598	2989616.2
16	7562667.896364	2793610.7	16	7487041.217400	2894114.0	16	7412170.805226	2994617.4
17	7558886.562416	2798611.9	17	7483297.696791	2899115.3	17	7408464.719824	2999618.7
18	7555107.119134	2803613.2	18	7479556.047943	2904116.5	18	7404760.487464	3004619.9
19	7551329.565575	2808614.4	19	7475816.269919	2909117.8	19	7401058.107220	3009621.2
20	7547553.900792	2813615.7	20	7472078.361784	2914119.0	20	7397357.578166	3014622.4

Column 30			Column 31			Column 32		
i	Number $c_{i,30}$	Logarithm $l_3(c_{i,30})$	i	Number $c_{i,31}$	Logarithm $l_3(c_{i,31})$	i	Number $c_{i,32}$	Logarithm $l_3(c_{i,32})$
0	7397003.733883	3015100.8	0	7323033.696544	3115604.1	0	7249803.359579	3216107.5
1	7393305.232016	3020102.0	1	7319372.179696	3120605.4	1	7246178.457899	3221108.7
2	7389608.579400	3025103.3	2	7315712.493606	3125606.6	2	7242555.368670	3226110.0
3	7385913.775110	3030104.5	3	7312054.637359	3130607.9	3	7238934.090985	3231111.2
4	7382220.818223	3035105.8	4	7308398.610040	3135609.1	4	7235314.623940	3236112.5
5	7378529.707813	3040107.0	5	7304744.410735	3140610.4	5	7231696.966628	3241113.7
6	7374840.442960	3045108.3	6	7301092.038530	3145611.6	6	7228081.118145	3246115.0
7	7371153.022738	3050109.5	7	7297441.492511	3150612.9	7	7224467.077586	3251116.2
8	7367467.446227	3055110.8	8	7293792.771764	3155614.1	8	7220854.844047	3256117.5
9	7363783.712504	3060112.0	9	7290145.875379	3160615.4	9	7217244.416625	3261118.7
10	7360101.820647	3065113.3	10	7286500.802441	3165616.6	10	7213635.794416	3266120.0
11	7356421.769737	3070114.5	11	7282857.552040	3170617.9	11	7210028.976519	3271121.2
12	7352743.558852	3075115.8	12	7279216.123264	3175619.1	12	7206423.962031	3276122.5
13	7349067.187073	3080117.0	13	7275576.515202	3180620.4	13	7202820.750050	3281123.7
14	7345392.653479	3085118.3	14	7271938.726944	3185621.6	14	7199219.339675	3286125.0
15	7341719.957152	3090119.5	15	7268302.757581	3190622.9	15	7195619.730005	3291126.2
16	7338049.097174	3095120.8	16	7264668.606202	3195624.1	16	7192021.920140	3296127.5
17	7334380.072625	3100122.0	17	7261036.271899	3200625.4	17	7188425.909180	3301128.7
18	7330712.882589	3105123.3	18	7257405.753763	3205626.6	18	7184831.696225	3306130.0
19	7327047.526148	3110124.5	19	7253777.050886	3210627.9	19	7181239.280377	3311131.2
20	7323384.002385	3115125.8	20	7250150.162361	3215629.1	20	7177648.660737	3316132.5

Column 33			Column 34			Column 35		
	Number	Logarithm		Number	Logarithm		Number	Logarithm
i	$c_{i,33}$	$l_3(c_{i,33})$	i	$c_{i,34}$	$l_3(c_{i,34})$	i	$c_{i,35}$	$l_3(c_{i,35})$
0	7177305.325983	3316610.8	0	7105532.272723	3417114.2	0	7034476.949996	3517617.5
1	7173716.673320	3321612.1	1	7101979.506587	3422115.4	1	7030959.711521	3522618.8
2	7170129.814983	3326613.3	2	7098428.516833	3427116.7	2	7027444.231665	3527620.0
3	7166544.750076	3331614.6	3	7094879.302575	3432117.9	3	7023930.509549	3532621.3
4	7162961.477701	3336615.8	4	7091331.862924	3437119.2	4	7020418.544294	3537622.6
5	7159379.996962	3341617.1	5	7087786.196992	3442120.4	5	7016908.335022	3542623.8
6	7155800.306963	3346618.3	6	7084242.303894	3447121.7	6	7013399.880855	3547625.1
7	7152222.406810	3351619.6	7	7080700.182742	3452122.9	7	7009893.180914	3552626.3
8	7148646.295606	3356620.8	8	7077159.832650	3457124.2	8	7006388.234324	3557627.6
9	7145071.972459	3361622.1	9	7073621.252734	3462125.4	9	7002885.040207	3562628.8
10	7141499.436472	3366623.3	10	7070084.442108	3467126.7	10	6999383.597687	3567630.1
11	7137928.686754	3371624.6	11	7066549.399887	3472127.9	11	6995883.905888	3572631.3
12	7134359.722411	3376625.8	12	7063016.125187	3477129.2	12	6992385.963935	3577632.6
13	7130792.542550	3381627.1	13	7059484.617124	3482130.4	13	6988889.770953	3582633.8
14	7127227.146278	3386628.3	14	7055954.874815	3487131.7	14	6985395.326067	3587635.1
15	7123663.532705	3391629.6	15	7052426.897378	3492132.9	15	6981902.628404	3592636.3
16	7120101.700939	3396630.8	16	7048900.683929	3497134.2	16	6978411.677090	3597637.6
17	7116541.650088	3401632.1	17	7045376.233587	3502135.4	17	6974922.471252	3602638.8
18	7112983.379263	3406633.3	18	7041853.545471	3507136.7	18	6971435.010016	3607640.1
19	7109426.887574	3411634.6	19	7038332.618698	3512137.9	19	6967949.292511	3612641.3
20	7105872.174130	3416635.8	20	7034813.452389	3517139.2	20	6964465.317865	3617642.6

Column 36			Column 37			Column 38		
	Number	Logarithm		Number	Logarithm		Number	Logarithm
i	$c_{i,36}$	$l_3(c_{i,36})$	i	$c_{i,37}$	$l_3(c_{i,37})$	i	$c_{i,38}$	$l_3(c_{i,38})$
0	6964132.180496	3618120.9	0	6894490.858691	3718624.3	0	6825545.950104	3819127.6
1	6960650.114405	3623122.2	1	6891043.613261	3723625.5	1	6822133.177129	3824128.9
2	6957169.789348	3628123.4	2	6887598.091455	3728626.8	2	6818722.110540	3829130.1
3	6953691.204454	3633124.7	3	6884154.292409	3733628.0	3	6815312.749485	3834131.4
4	6950214.358851	3638125.9	4	6880712.215263	3738629.3	4	6811905.093110	3839132.6
5	6946739.251672	3643127.2	5	6877271.859155	3743630.5	5	6808499.140564	3844133.9
6	6943265.882046	3648128.4	6	6873833.223226	3748631.8	6	6805094.890993	3849135.1
7	6939794.249105	3653129.7	7	6870396.306614	3753633.0	7	6801692.343548	3854136.4
8	6936324.351981	3658130.9	8	6866961.108461	3758634.3	8	6798291.497376	3859137.6
9	6932856.189805	3663132.2	9	6863527.627907	3763635.5	9	6794892.351627	3864138.9
10	6929389.761710	3668133.4	10	6860095.864093	3768636.8	10	6791494.905452	3869140.1
11	6925925.066829	3673134.7	11	6856665.816161	3773638.0	11	6788099.157999	3874141.4
12	6922462.104295	3678135.9	12	6853237.483252	3778639.3	12	6784705.108420	3879142.6
13	6919000.873243	3683137.2	13	6849810.864511	3783640.5	13	6781312.755866	3884143.9
14	6915541.372807	3688138.4	14	6846385.959079	3788641.8	14	6777922.099488	3889145.1
15	6912083.602120	3693139.7	15	6842962.766099	3793643.0	15	6774533.138438	3894146.4
16	6908627.560319	3698140.9	16	6839541.284716	3798644.3	16	6771145.871869	3899147.6
17	6905173.246539	3703142.2	17	6836121.514074	3803645.5	17	6767760.298933	3904148.9
18	6901720.659916	3708143.4	18	6832703.453317	3808646.8	18	6764376.418783	3909150.1
19	6898269.799586	3713144.7	19	6829287.101590	3813648.0	19	6760994.230574	3914151.4
20	6894820.664686	3718145.9	20	6825872.458039	3818649.3	20	6757613.733459	3919152.6

Column 39			Column 40			Column 41		
i	Number $c_{i,39}$	Logarithm $l_3(c_{i,39})$	i	Number $c_{i,40}$	Logarithm $l_3(c_{i,40})$	i	Number $c_{i,41}$	Logarithm $l_3(c_{i,41})$
0	6757290.490603	3919631.0	0	6689717.585697	4020134.3	0	6622820.409840	4120637.7
1	6753911.845358	3924632.2	1	6686372.726904	4025135.6	1	6619508.999635	4125639.0
2	6750534.889435	3929633.5	2	6683029.540541	4030136.8	2	6616199.245135	4130640.2
3	6747159.621990	3934634.7	3	6679688.025770	4035138.1	3	6612891.145513	4135641.5
4	6743786.042179	3939636.0	4	6676348.181757	4040139.3	4	6609584.699940	4140642.7
5	6740414.149158	3944637.2	5	6673010.007666	4045140.6	5	6606279.907590	4145644.0
6	6737043.942083	3949638.5	6	6669673.502663	4050141.8	6	6602976.767636	4150645.2
7	6733675.420112	3954639.7	7	6666338.665911	4055143.1	7	6599675.279252	4155646.5
8	6730308.582402	3959641.0	8	6663005.496578	4060144.3	8	6596375.441613	4160647.7
9	6726943.428111	3964642.2	9	6659673.993830	4065145.6	9	6593077.253892	4165649.0
10	6723579.956397	3969643.5	10	6656344.156833	4070146.8	10	6589780.715265	4170650.2
11	6720218.166419	3974644.7	11	6653015.984755	4075148.1	11	6586485.824907	4175651.5
12	6716858.057336	3979646.0	12	6649689.476762	4080149.3	12	6583192.581995	4180652.7
13	6713499.628307	3984647.2	13	6646364.632024	4085150.6	13	6579900.985704	4185654.0
14	6710142.878493	3989648.5	14	6643041.449708	4090151.8	14	6576611.035211	4190655.2
15	6706787.807054	3994649.7	15	6639719.928983	4095153.1	15	6573322.729693	4195656.5
16	6703434.413150	3999651.0	16	6636400.069019	4100154.3	16	6570036.068328	4200657.7
17	6700082.695944	4004652.2	17	6633081.868984	4105155.6	17	6566751.050294	4205659.0
18	6696732.654596	4009653.5	18	6629765.328050	4110156.8	18	6563467.674769	4210660.2
19	6693384.288268	4014654.7	19	6626450.445386	4115158.1	19	6560185.940932	4215661.5
20	6690037.596124	4019656.0	20	6623137.220163	4120159.3	20	6556905.847961	4220662.7

Column 42			Column 43			Column 44		
	Number	Logarithm		Number	Logarithm		Number	Logarithm
i	$c_{i,42}$	$l_3(c_{i,42})$	i	$c_{i,43}$	$l_3(c_{i,43})$	i	$c_{i,44}$	$l_3(c_{i,44})$
0	6556592.205741	4221141.1	0	6491026.283684	4321644.4	0	6426116.020847	4422147.8
1	6553313.909639	4226142.3	1	6487780.770542	4326645.7	1	6422902.962837	4427149.0
2	6550037.252684	4231143.6	2	6484536.880157	4331646.9	2	6419691.511355	4432150.3
3	6546762.234057	4236144.8	3	6481294.611717	4336648.2	3	6416481.665600	4437151.5
4	6543488.852940	4241146.1	4	6478053.964411	4341649.4	4	6413273.424767	4442152.8
5	6540217.108514	4246147.3	5	6474814.937429	4346650.7	5	6410066.788054	4447154.0
6	6536946.999960	4251148.6	6	6471577.529960	4351651.9	6	6406861.754660	4452155.3
7	6533678.526460	4256149.8	7	6468341.741195	4356653.2	7	6403658.323783	4457156.5
8	6530411.687196	4261151.1	8	6465107.570324	4361654.4	8	6400456.494621	4462157.8
9	6527146.481353	4266152.3	9	6461875.016539	4366655.7	9	6397256.266374	4467159.0
10	6523882.908112	4271153.6	10	6458644.079031	4371656.9	10	6394057.638241	4472160.3
11	6520620.966658	4276154.8	11	6455414.756992	4376658.2	11	6390860.609422	4477161.5
12	6517360.656175	4281156.1	12	6452187.049613	4381659.4	12	6387665.179117	4482162.8
13	6514101.975847	4286157.3	13	6448960.956088	4386660.7	13	6384471.346527	4487164.0
14	6510844.924859	4291158.6	14	6445736.475610	4391661.9	14	6381279.110854	4492165.3
15	6507589.502396	4296159.8	15	6442513.607372	4396663.2	15	6378088.471299	4497166.5
16	6504335.707645	4301161.1	16	6439292.350569	4401664.4	16	6374899.427063	4502167.8
17	6501083.539791	4306162.3	17	6436072.704393	4406665.7	17	6371711.977349	4507169.0
18	6497832.998021	4311163.6	18	6432854.668041	4411666.9	18	6368526.121361	4512170.3
19	6494584.081522	4316164.8	19	6429638.240707	4416668.2	19	6365341.858300	4517171.5
20	6491336.789482	4321166.1	20	6426423.421587	4421669.4	20	6362159.187371	4522172.8

Column 45			Column 46			Column 47		
i	Number $c_{i,45}$	Logarithm $l_3(c_{i,45})$	i	Number $c_{i,46}$	Logarithm $l_3(c_{i,46})$	i	Number $c_{i,47}$	Logarithm $l_3(c_{i,47})$
0	6361854.860639	4522651.1	0	6298236.312032	4623154.5	0	6235253.948912	4723657.9
1	6358673.933208	4527652.4	1	6295087.193876	4628155.7	1	6232136.321938	4728659.1
2	6355494.596242	4532653.6	2	6291939.650279	4633157.0	2	6229020.253777	4733660.4
3	6352316.848944	4537654.9	3	6288793.680454	4638158.2	3	6225905.743650	4738661.6
4	6349140.690519	4542656.1	4	6285649.283614	4643159.5	4	6222792.790778	4743662.9
5	6345966.120174	4547657.4	5	6282506.458972	4648160.7	5	6219681.394382	4748664.1
6	6342793.137114	4552658.6	6	6279365.205743	4653162.0	6	6216571.553685	4753665.4
7	6339621.740545	4557659.9	7	6276225.523140	4658163.2	7	6213463.267908	4758666.6
8	6336451.929675	4562661.1	8	6273087.410378	4663164.5	8	6210356.536274	4763667.9
9	6333283.703710	4567662.4	9	6269950.866673	4668165.7	9	6207251.358006	4768669.1
10	6330117.061858	4572663.6	10	6266815.891240	4673167.0	10	6204147.732327	4773670.4
11	6326952.003327	4577664.9	11	6263682.483294	4678168.2	11	6201045.658461	4778671.6
12	6323788.527326	4582666.1	12	6260550.642052	4683169.5	12	6197945.135632	4783672.9
13	6320626.633062	4587667.4	13	6257420.366731	4688170.7	13	6194846.163064	4788674.1
14	6317466.319746	4592668.6	14	6254291.656548	4693172.0	14	6191748.739983	4793675.4
15	6314307.586586	4597669.9	15	6251164.510720	4698173.2	15	6188652.865613	4798676.6
16	6311150.432792	4602671.1	16	6248038.928464	4703174.5	16	6185558.539180	4803677.9
17	6307994.857576	4607672.4	17	6244914.909000	4708175.7	17	6182465.759910	4808679.1
18	6304840.860147	4612673.6	18	6241792.451546	4713177.0	18	6179374.527030	4813680.4
19	6301688.439717	4617674.9	19	6238671.555320	4718178.3	19	6176284.839767	4818681.6
20	6298537.595497	4622676.1	20	6235552.219542	4723179.5	20	6173196.697347	4823682.9

Column 48			Column 49			Column 50		
	Number	Logarithm		Number	Logarithm		Number	Logarithm
i	$c_{i,48}$	$l_3(c_{i,48})$	i	$c_{i,49}$	$l_3(c_{i,49})$	i	$c_{i,50}$	$l_3(c_{i,50})$
0	6172901.409423	4824161.2	0	6111172.395329	4924664.6	0	6050060.671375	5025167.9
1	6169814.958718	4829162.5	1	6108116.809131	4929665.8	1	6047035.641040	5030169.2
2	6166730.051239	4834163.7	2	6105062.750726	4934667.1	2	6044012.123219	5035170.4
3	6163646.686213	4839165.0	3	6102010.219351	4939668.3	3	6040990.117158	5040171.7
4	6160564.862870	4844166.2	4	6098959.214241	4944669.6	4	6037969.622099	5045172.9
5	6157484.580439	4849167.5	5	6095909.734634	4949670.8	5	6034950.637288	5050174.2
6	6154405.838148	4854168.7	6	6092861.779767	4954672.1	6	6031933.161969	5055175.4
7	6151328.635229	4859170.0	7	6089815.348877	4959673.3	7	6028917.195388	5060176.7
8	6148252.970912	4864171.2	8	6086770.441203	4964674.6	8	6025902.736791	5065177.9
9	6145178.844426	4869172.5	9	6083727.055982	4969675.8	9	6022889.785422	5070179.2
10	6142106.255004	4874173.7	10	6080685.192454	4974677.1	10	6019878.340529	5075180.4
11	6139035.201877	4879175.0	11	6077644.849858	4979678.3	11	6016868.401359	5080181.7
12	6135965.684276	4884176.2	12	6074606.027433	4984679.6	12	6013859.967159	5085182.9
13	6132897.701433	4889177.5	13	6071568.724419	4989680.8	13	6010853.037175	5090184.2
14	6129831.252583	4894178.7	14	6068532.940057	4994682.1	14	6007847.610656	5095185.4
15	6126766.336956	4899180.0	15	6065498.673587	4999683.3	15	6004843.686851	5100186.7
16	6123702.953788	4904181.2	16	6062465.924250	5004684.6	16	6001841.265008	5105187.9
17	6120641.102311	4909182.5	17	6059434.691288	5009685.8	17	5998840.344375	5110189.2
18	6117580.781760	4914183.7	18	6056404.973942	5014687.1	18	5995840.924203	5115190.4
19	6114521.991369	4919185.0	19	6053376.771455	5019688.3	19	5992843.003741	5120191.7
20	6111464.730373	4924186.2	20	6050350.083070	5024689.6	20	5989846.582239	5125192.9

Column 51			Column 52			Column 53		
i	Number $c_{i,51}$	Logarithm $l_3(c_{i,51})$	i	Number $c_{i,52}$	Logarithm $l_3(c_{i,52})$	i	Number $c_{i,53}$	Logarithm $l_3(c_{i,53})$
0	5989560.064662	5125671.3	0	5929664.464015	5226174.6	0	5870367.819375	5326678.0
1	5986565.284629	5130672.5	1	5926699.631783	5231175.9	1	5867432.635465	5331679.3
2	5983572.001987	5135673.8	2	5923736.281967	5236177.1	2	5864498.919147	5336680.5
3	5980580.215986	5140675.0	3	5920774.413826	5241178.4	3	5861566.669688	5341681.8
4	5977589.925878	5145676.3	4	5917814.026619	5246179.6	4	5858635.886353	5346683.0
5	5974601.130915	5150677.5	5	5914855.119606	5251180.9	5	5855706.568410	5351684.3
6	5971613.830350	5155678.8	6	5911897.692046	5256182.1	6	5852778.715126	5356685.5
7	5968628.023434	5160680.0	7	5908941.743200	5261183.4	7	5849852.325768	5361686.8
8	5965643.709423	5165681.3	8	5905987.272328	5266184.6	8	5846927.399605	5366688.0
9	5962660.887568	5170682.5	9	5903034.278692	5271185.9	9	5844003.935905	5371689.3
10	5959679.557124	5175683.8	10	5900082.761553	5276187.1	10	5841081.933937	5376690.5
11	5956699.717346	5180685.0	11	5897132.720172	5281188.4	11	5838161.392970	5381691.8
12	5953721.367487	5185686.3	12	5894184.153812	5286189.6	12	5835242.312274	5386693.0
13	5950744.506803	5190687.5	13	5891237.061735	5291190.9	13	5832324.691118	5391694.3
14	5947769.134550	5195688.8	14	5888291.443204	5296192.1	14	5829408.528772	5396695.5
15	5944795.249983	5200690.0	15	5885347.297483	5301193.4	15	5826493.824508	5401696.8
16	5941822.852358	5205691.3	16	5882404.623834	5306194.7	16	5823580.577596	5406698.0
17	5938851.940931	5210692.5	17	5879463.421522	5311195.9	17	5820668.787307	5411699.3
18	5935882.514961	5215693.8	18	5876523.689811	5316197.2	18	5817758.452913	5416700.5
19	5932914.573703	5220695.0	19	5873585.427966	5321198.4	19	5814849.573687	5421701.8
20	5929948.116417	5225696.3	20	5870648.635252	5326199.7	20	5811942.148900	5426703.0

Column 54			Column 55			Column 56		
i	Number $c_{i,54}$	Logarithm $l_3(c_{i,54})$	i	Number $c_{i,55}$	Logarithm $l_3(c_{i,55})$	i	Number $c_{i,56}$	Logarithm $l_3(c_{i,56})$
0	5811664.141181	5427181.4	0	5753547.499769	5527684.7	0	5696012.024772	5628188.1
1	5808758.309111	5432182.6	1	5750670.726019	5532686.0	1	5693164.018759	5633189.3
2	5805853.929956	5437183.9	2	5747795.390656	5537687.2	2	5690317.436750	5638190.6
3	5802951.002991	5442185.1	3	5744921.492961	5542688.5	3	5687472.278031	5643191.8
4	5800049.527489	5447186.4	4	5742049.032215	5547689.7	4	5684628.541892	5648193.1
5	5797149.502726	5452187.6	5	5739178.007698	5552691.0	5	5681786.227621	5653194.3
6	5794250.927974	5457188.9	6	5736308.418695	5557692.2	6	5678945.334508	5658195.6
7	5791353.802510	5462190.1	7	5733440.264485	5562693.5	7	5676105.861840	5663196.8
8	5788458.125609	5467191.4	8	5730573.544353	5567694.7	8	5673267.808910	5668198.1
9	5785563.896546	5472192.6	9	5727708.257581	5572696.0	9	5670431.175005	5673199.3
10	5782671.114598	5477193.9	10	5724844.403452	5577697.2	10	5667595.959418	5678200.6
11	5779779.779041	5482195.1	11	5721981.981250	5582698.5	11	5664762.161438	5683201.8
12	5776889.889151	5487196.4	12	5719120.990260	5587699.7	12	5661929.780357	5688203.1
13	5774001.444207	5492197.6	13	5716261.429765	5592701.0	13	5659098.815467	5693204.3
14	5771114.443485	5497198.9	14	5713403.299050	5597702.2	14	5656269.266059	5698205.6
15	5768228.886263	5502200.1	15	5710546.597400	5602703.5	15	5653441.131426	5703206.8
16	5765344.771820	5507201.4	16	5707691.324101	5607704.7	16	5650614.410860	5708208.1
17	5762462.099434	5512202.6	17	5704837.478439	5612706.0	17	5647789.103655	5713209.3
18	5759580.868384	5517203.9	18	5701985.059700	5617707.2	18	5644965.209103	5718210.6
19	5756701.077950	5522205.1	19	5699134.067170	5622708.5	19	5642142.726499	5723211.8
20	5753822.727411	5527206.4	20	5696284.500137	5627709.7	20	5639321.655135	5728213.1

Column 57			Column 58			Column 59		
i	Number $c_{i,57}$	Logarithm $l_3(c_{i,57})$	i	Number $c_{i,58}$	Logarithm $l_3(c_{i,58})$	i	Number $c_{i,59}$	Logarithm $l_3(c_{i,59})$
0	5639051.904524	5728691.4	0	5582661.385479	5829194.8	0	5526834.771624	5929698.2
1	5636232.378572	5733692.7	1	5579870.054786	5834196.0	1	5524071.354238	5934699.4
2	5633414.262382	5738693.9	2	5577080.119759	5839197.3	2	5521309.318561	5939700.7
3	5630597.555251	5743695.2	3	5574291.579699	5844198.5	3	5518548.663902	5944701.9
4	5627782.256474	5748696.4	4	5571504.433909	5849199.8	4	5515789.389570	5949703.2
5	5624968.365345	5753697.7	5	5568718.681692	5854201.0	5	5513031.494875	5954704.4
6	5622155.881163	5758698.9	6	5565934.322351	5859202.3	6	5510274.979127	5959705.7
7	5619344.803222	5763700.2	7	5563151.355190	5864203.5	7	5507519.841638	5964706.9
8	5616535.130820	5768701.4	8	5560369.779512	5869204.8	8	5504766.081717	5969708.2
9	5613726.863255	5773702.7	9	5557589.594622	5874206.0	9	5502013.698676	5974709.4
10	5610919.999823	5778703.9	10	5554810.799825	5879207.3	10	5499262.691827	5979710.7
11	5608114.539823	5783705.2	11	5552033.394425	5884208.5	11	5496513.060481	5984711.9
12	5605310.482554	5788706.4	12	5549257.377728	5889209.8	12	5493764.803951	5989713.2
13	5602507.827312	5793707.7	13	5546482.749039	5894211.1	13	5491017.921549	5994714.4
14	5599706.573399	5798708.9	14	5543709.507665	5899212.3	14	5488272.412588	5999715.7
15	5596906.720112	5803710.2	15	5540937.652911	5904213.6	15	5485528.276382	6004716.9
16	5594108.266752	5808711.4	16	5538167.184084	5909214.8	16	5482785.512244	6009718.2
17	5591311.212618	5813712.7	17	5535398.100492	5914216.1	17	5480044.119487	6014719.4
18	5588515.557012	5818713.9	18	5532630.401442	5919217.3	18	5477304.097428	6019720.7
19	5585721.299234	5823715.2	19	5529864.086241	5924218.6	19	5474565.445379	6024721.9
20	5582928.438584	5828716.4	20	5527099.154198	5929219.8	20	5471828.162656	6029723.2

Column 60		
i	Number $c_{i,60}$	Logarithm $l_3(c_{i,60})$
0	5471566.423908	6030201.5
1	5468830.640696	6035202.8
2	5466096.225375	6040204.0
3	5463363.177263	6045205.3
4	5460631.495674	6050206.5
5	5457901.179926	6055207.8
6	5455172.229336	6060209.0
7	5452444.643222	6065210.3
8	5449718.420900	6070211.5
9	5446993.561689	6075212.8
10	5444270.064909	6080214.0
11	5441547.929876	6085215.3
12	5438827.155911	6090216.5
13	5436107.742333	6095217.8
14	5433389.688462	6100219.0
15	5430672.993618	6105220.3
16	5427957.657121	6110221.5
17	5425243.678293	6115222.8
18	5422531.056453	6120224.0
19	5419819.790925	6125225.3
20	5417109.881030	6130226.5

Column 61		
i	Number $c_{i,61}$	Logarithm $l_3(c_{i,61})$
0	5416850.759669	6130704.9
1	5414142.334289	6135706.1
2	5411435.263122	6140707.4
3	5408729.545490	6145708.6
4	5406025.180717	6150709.9
5	5403322.168127	6155711.1
6	5400620.507043	6160712.4
7	5397920.196789	6165713.6
8	5395221.236691	6170714.9
9	5392523.626073	6175716.1
10	5389827.364260	6180717.4
11	5387132.450577	6185718.6
12	5384438.884352	6190719.9
13	5381746.664910	6195721.1
14	5379055.791577	6200722.4
15	5376366.263682	6205723.6
16	5373678.080550	6210724.9
17	5370991.241510	6215726.1
18	5368305.745889	6220727.4
19	5365621.593016	6225728.6
20	5362938.782219	6230729.9

Column 62		
i	Number $c_{i,62}$	Logarithm $l_3(c_{i,62})$
0	5362682.252072	6231208.2
1	5360000.910946	6236209.5
2	5357320.910490	6241210.7
3	5354642.250035	6246212.0
4	5351964.928910	6251213.2
5	5349288.946446	6256214.5
6	5346614.301972	6261215.7
7	5343940.994821	6266217.0
8	5341269.024324	6271218.2
9	5338598.389812	6276219.5
10	5335929.090617	6281220.7
11	5333261.126072	6286222.0
12	5330594.495509	6291223.2
13	5327929.198261	6296224.5
14	5325265.233662	6301225.7
15	5322602.601045	6306227.0
16	5319941.299744	6311228.2
17	5317281.329094	6316229.5
18	5314622.688430	6321230.7
19	5311965.377086	6326232.0
20	5309309.394397	6331233.2

Column 63			Column 64			Column 65		
i	Number $c_{i,63}$	Logarithm $l_3(c_{i,63})$	i	Number $c_{i,64}$	Logarithm $l_3(c_{i,64})$	i	Number $c_{i,65}$	Logarithm $l_3(c_{i,65})$
0	5309055.429551	6331711.6	0	5255964.875256	6432214.9	0	5203405.226503	6532718.3
1	5306400.901836	6336712.8	1	5253336.892818	6437216.2	1	5200803.523890	6537719.6
2	5303747.701385	6341714.1	2	5250710.224372	6442217.4	2	5198203.122128	6542720.8
3	5301095.827535	6346715.3	3	5248084.869259	6447218.7	3	5195604.020567	6547722.1
4	5298445.279621	6351716.6	4	5245460.826825	6452219.9	4	5193006.218557	6552723.3
5	5295796.056981	6356717.8	5	5242838.096411	6457221.2	5	5190409.715447	6557724.6
6	5293148.158953	6361719.1	6	5240216.677363	6462222.4	6	5187814.510590	6562725.8
7	5290501.584873	6366720.3	7	5237596.569024	6467223.7	7	5185220.603334	6567727.1
8	5287856.334081	6371721.6	8	5234977.770740	6472224.9	8	5182627.993033	6572728.3
9	5285212.405914	6376722.8	9	5232360.281855	6477226.2	9	5180036.679036	6577729.6
10	5282569.799711	6381724.1	10	5229744.101714	6482227.5	10	5177446.660697	6582730.8
11	5279928.514811	6386725.3	11	5227129.229663	6487228.7	11	5174857.937366	6587732.1
12	5277288.550554	6391726.6	12	5224515.665048	6492230.0	12	5172270.508397	6592733.3
13	5274649.906278	6396727.8	13	5221903.407215	6497231.2	13	5169684.373143	6597734.6
14	5272012.581325	6401729.1	14	5219292.455512	6502232.5	14	5167099.530957	6602735.8
15	5269376.575034	6406730.3	15	5216682.809284	6507233.7	15	5164515.981191	6607737.1
16	5266741.886747	6411731.6	16	5214074.467879	6512235.0	16	5161933.723201	6612738.3
17	5264108.515804	6416732.8	17	5211467.430646	6517236.2	17	5159352.756339	6617739.6
18	5261476.461546	6421734.1	18	5208861.696930	6522237.5	18	5156773.079961	6622740.8
19	5258845.723315	6426735.3	19	5206257.266082	6527238.7	19	5154194.693421	6627742.1
20	5256216.300453	6431736.6	20	5203654.137449	6532240.0	20	5151617.596074	6632743.3

Column 66			Column 67			Column 68		
i	Number $c_{i,66}$	Logarithm $l_3(c_{i,66})$	i	Number $c_{i,67}$	Logarithm $l_3(c_{i,67})$	i	Number $c_{i,68}$	Logarithm $l_3(c_{i,68})$
0	5151371.174238	6633221.7	0	5099857.462496	6733725.0	0	5048858.887871	6834228.4
1	5148795.488651	6638222.9	1	5097307.533764	6738726.3	1	5046334.458427	6839229.6
2	5146221.090907	6643224.2	2	5094758.879998	6743727.5	2	5043811.291198	6844230.9
3	5143647.980361	6648225.4	3	5092211.500558	6748728.8	3	5041289.385552	6849232.1
4	5141076.156371	6653226.7	4	5089665.394807	6753730.0	4	5038768.740859	6854233.4
5	5138505.618293	6658227.9	5	5087120.562110	6758731.3	5	5036249.356489	6859234.6
6	5135936.365484	6663229.2	6	5084577.001829	6763732.5	6	5033731.231811	6864235.9
7	5133368.397301	6668230.4	7	5082034.713328	6768733.8	7	5031214.366195	6869237.1
8	5130801.713102	6673231.7	8	5079493.695971	6773735.0	8	5028698.759011	6874238.4
9	5128236.312246	6678232.9	9	5076953.949123	6778736.3	9	5026184.409632	6879239.6
10	5125672.194090	6683234.2	10	5074415.472149	6783737.5	10	5023671.317427	6884240.9
11	5123109.357993	6688235.4	11	5071878.264413	6788738.8	11	5021159.481768	6889242.1
12	5120547.803314	6693236.7	12	5069342.325280	6793740.0	12	5018648.902028	6894243.4
13	5117987.529412	6698237.9	13	5066807.654118	6798741.3	13	5016139.577577	6899244.6
14	5115428.535647	6703239.2	14	5064274.250291	6803742.5	14	5013631.507788	6904245.9
15	5112870.821379	6708240.4	15	5061742.113166	6808743.8	15	5011124.692034	6909247.1
16	5110314.385969	6713241.7	16	5059211.242109	6813745.0	16	5008619.129688	6914248.4
17	5107759.228776	6718242.9	17	5056681.636488	6818746.3	17	5006114.820123	6919249.6
18	5105205.349161	6723244.2	18	5054153.295670	6823747.5	18	5003611.762713	6924250.9
19	5102652.746487	6728245.4	19	5051626.219022	6828748.8	19	5001109.956832	6929252.1
20	5100101.420113	6733246.7	20	5049100.405912	6833750.0	20	4998609.401853	6934253.4