10

VECTORS

10.1 VECTOR ALGEBRA

Figure 10.1.1 shows a directed line segment from the point P to the point Q.

Figure 10.1.1

We pictorially represent a directed line segment as an arrow from P to Q, and use the symbol $\overrightarrow{PQ}$. Mathematically, a directed line segment is most easily represented as an ordered pair of points.

The *directed line segment* from P to Q, in symbols $\overrightarrow{PQ}$, is the ordered pair of points (P, Q). P is called the *initial point*, and Q, the *terminal point*, of the directed line segment.

The directed line segments $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ are considered to be different. $\overrightarrow{QP}$ has initial point Q and terminal point P. If $P(p_1, p_2)$ and $Q(q_1, q_2)$ are two points in the plane, the *x-component* of $\overrightarrow{PQ}$ is the increment $q_1 - p_1$ of x from P to Q, and the *y-component* is the increment $q_2 - p_2$ of y, as shown in Figure 10.1.2.

$$x\text{-component of } \overrightarrow{PQ} = q_1 - p_1.$$
$$y\text{-component of } \overrightarrow{PQ} = q_2 - p_2.$$

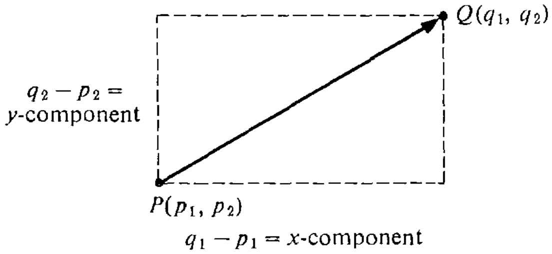


Figure 10.1.2

Usually we are not really interested in the exact placement of a directed line segment $\overrightarrow{PQ}$ on the (x, y) plane, but in the length and direction of $\overrightarrow{PQ}$. These can be determined by the x and y components of $\overrightarrow{PQ}$. We are thus led to the notion of a vector.

DEFINITION

*The family of all directed line segments with the same components as $\overrightarrow{PQ}$ will be called the **vector** from P to Q. We say that $\overrightarrow{PQ}$ **represents** this vector.*

Since all directed line segments with the same components have the same length and direction, a vector may be regarded as a quantity which has length and direction.

Vectors arise quite naturally in both physics and economics. Here are some examples of vector quantities.

Position If an object is at the point (p_1, p_2) in the plane, its position vector is the vector with components p_1 and p_2.

Velocity If a particle is moving in the plane according to the parametric equations

$$x = f(t), \qquad y = g(t),$$

the velocity vector is the vector with x and y components dx/dt and dy/dt.

Acceleration The acceleration vector of a moving particle has the x and y components d^2x/dt^2 and d^2y/dt^2.

Force In physics, force is a vector quantity which will accelerate a free particle in the direction of the force vector at a rate proportional to the length of the force vector

Displacement (change in position) If an object moves from the point P to the point Q, its displacement vector is the vector from P to Q.

Commodity vector In economics, one often compares two or more commodities (such as guns and butter). If a trader in a market has a quantity a_1 of one commodity and a_2 of another, his commodity vector has the x and y components (a_1, a_2).

Price vector If two commodities have prices p_1 and p_2 respectively, the price vector has components (p_1, p_2). The components of a commodity or price vector are always greater than or equal to zero.

EXAMPLE 1 Find the components of the vectors represented by the given directed line segments.

(a) $\overrightarrow{(3, 2), (5, 1)}$.

$$x\text{-component} = 5 - 3 = 2, \qquad y\text{-component} = 1 - 2 = -1.$$

(b) $\overrightarrow{(0, -2), (2, -3)}$.

$$x\text{-component} = 2 - 0 = 2, \qquad y\text{-component} = -3 - (-2) = -1.$$

Notice that both of these directed line segments represent the same vector. They are shown in Figure 10.1.3.

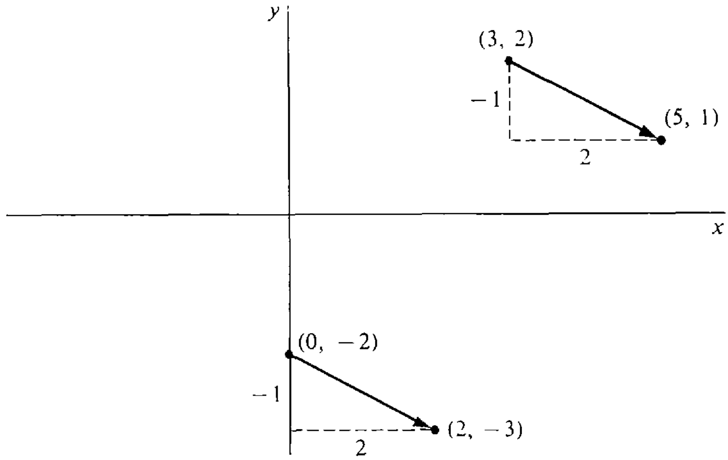


Figure 10.1.3

Vectors are denoted by boldface letters, **A**. A vector is represented by a whole family of directed line segments. However, given a vector **A** and an initial point P, there is exactly one point Q such that the directed line segment $\overrightarrow{PQ}$ represents **A**. To find the x-coordinate of Q we add the x-coordinate of P and the x component of **A**; similarly for the y-coordinate.

EXAMPLE 2 Let **A** be the vector with components -4 and 1, and let P be the point $(1, 2)$. Find Q so that $\overrightarrow{PQ}$ represents **A**.

Q has the x-coordinate $1 + (-4) = -3$ and the y-coordinate $2 + 1 = 3$. Thus $Q = (-3, 3)$, as shown in Figure 10.1.4.

We shall now begin the algebra of vectors. In vector algebra, real numbers are called *scalars*. We study two different kinds of quantities, scalars and vectors.

The *length* (or norm) of a vector **A** is the distance between P and Q where $\overrightarrow{PQ}$ represents **A**. The length is a scalar, denoted by $|\mathbf{A}|$. If **A** has components a_1 and a_2, then the length, shown in Figure 10.1.5, is given by the distance formula,

$$|\mathbf{A}| = \sqrt{a_1^2 + a_2^2}.$$

The length of a position vector is the *distance from the origin*. The length of a

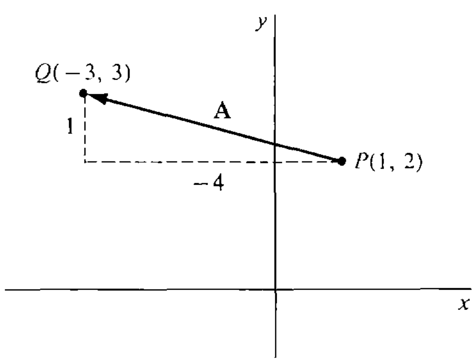


Figure 10.1.4

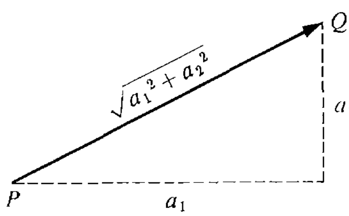


Figure 10.1.5 Length of a Vector

velocity vector is the *speed* of a particle. The length of a force vector is the *magnitude* of the force. The length of a displacement vector is the *distance moved.* For price or commodity vectors, the notion of length does not arise in a natural way.

EXAMPLE 3 The vector **A** with components 3 and −4 has length $|\mathbf{A}| = \sqrt{3^2 + (-4)^2} = 5$.

The vector with components (0, 0) is called the *zero vector*, denoted by **0**. The zero vector is represented by the degenerate line segments $\overrightarrow{PP}$. It has no direction. The length of the zero vector is zero, while the length of every other vector is a positive scalar.

The sum **A** + **B** of vectors **A** and **B** is defined as follows. Let $\overrightarrow{PQ}$ represent **A** and let $\overrightarrow{QR}$ represent **B**. Then **A** + **B** is the vector represented by $\overrightarrow{PR}$. More briefly, if **A** is the vector from P to Q and **B** is the vector from Q to R, then **A** + **B** is the vector from P to R. Figure 10.1.6 shows two ways of drawing the sum **A** + **B**.

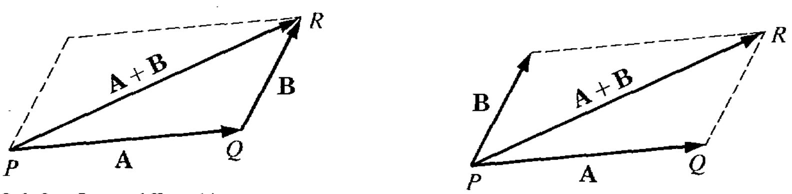
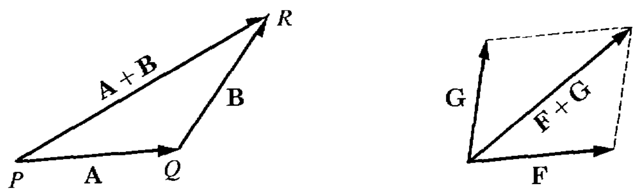


Figure 10.1.6 Sum of Two Vectors

If an object in the plane originally has the position vector **P** and is moved by a displacement vector **D**, its new position vector will be the vector sum **P** + **D**. If an object is moved twice, first by a displacement vector **D** and then by a displacement vector **E**, the total displacement vector is the sum **D** + **E**.

If two forces **F** and **G** are acting simultaneously on an object, their combined effect is the vector sum **F** + **G** (Figure 10.1.7). The combined effect of three or more forces acting on an object is also the vector sum, e.g., (**F** + **G**) + **H**. *Newton's first law of motion* states that if an object is at rest, the vector sum of all forces acting on the object is the zero vector.



Changes in position

Forces

Figure 10.1.7

In economics, if a trader initially has a commodity vector **A** and buys a commodity vector **B** (i.e., he buys a quantity b_1 of commodity one and b_2 of commodity two), his new commodity vector will be the vector sum **A** + **B**.

The vector sum is also useful in discussing an exchange between two or more traders. Suppose traders A and B initially have commodity vectors $\mathbf{A}_1$ and $\mathbf{B}_1$. After exchanging goods, they have new commodity vectors $\mathbf{A}_2$ and $\mathbf{B}_2$. Since the total amount of each good remains unchanged, we see that $\mathbf{A}_1 + \mathbf{B}_1 = \mathbf{A}_2 + \mathbf{B}_2$.

Vector sums obey rules similar to the rules for sums of real numbers.

THEOREM 1

Let **A**, **B**, *and* **C** *be vectors.*

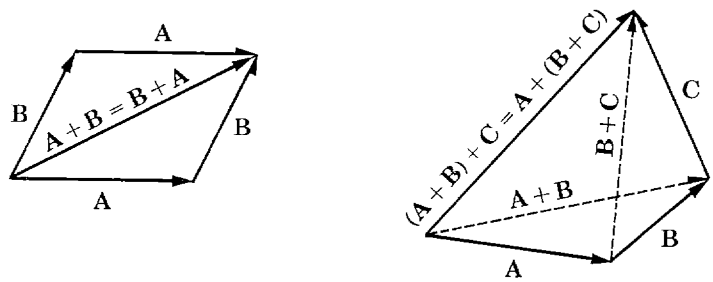
(i) Identity Law $\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$.

(ii) Commutative Law $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

(iii) Associative Law $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.

(iv) Triangle Inequality $|\mathbf{A} + \mathbf{B}| \leq |\mathbf{A}| + |\mathbf{B}|$.

We shall skip the proofs, which use the corresponding laws for real numbers. The Commutative and Associative Laws are illustrated by Figure 10.1.8.

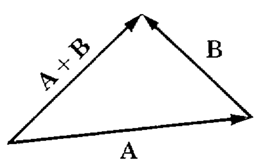


Commutative law   Associative law

Figure 10.1.8

The Triangle Inequality says that the length of one side of a triangle is at most the sum of the lengths of the other two sides. This is because the vectors **A**, **B**, and **A** + **B** are represented by sides of a triangle. The proof of the Triangle Inequality is left as a problem (with a hint). It is illustrated in Figure 10.1.9.

The sum of three or more vectors is formed in the same way as the sum of two vectors, as in Figure 10.1.10.



Triangle inequality

Figure 10.1.9

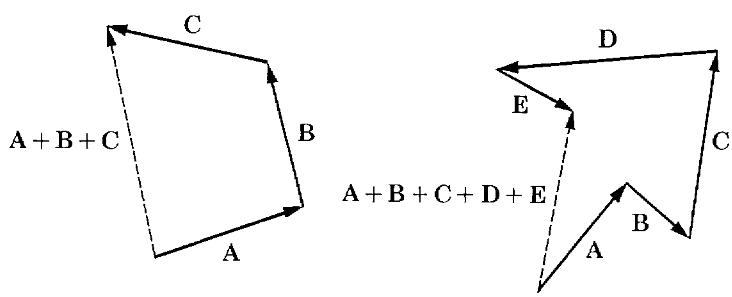


Figure 10.1.10 Sum of Vectors

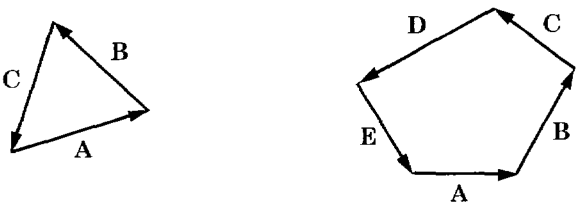


Figure 10.1.11 $\mathbf{A}+\mathbf{B}+\mathbf{C}=\mathbf{O}$ $\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D}+\mathbf{E}=\mathbf{O}$

Vector negative

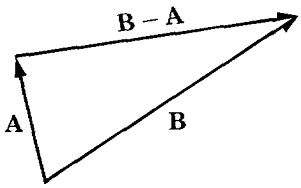
Figure 10.1.12

The sum of the vectors (clockwise or counterclockwise) around the perimeter of a triangle or polygon is always the zero vector (Figure 10.1.11).

We next define the *vector negative*, $-\mathbf{A}$, and the *vector difference*, $\mathbf{B}-\mathbf{A}$. If $\mathbf{A}$ is the vector from P to Q, then $-\mathbf{A}$ is the vector from Q to P (Figure 10.1.12). $\mathbf{B}-\mathbf{A}$ is the vector which, when added to $\mathbf{A}$, gives $\mathbf{B}$; i.e.,

$$\mathbf{A}+(\mathbf{B}-\mathbf{A})=\mathbf{B}.$$

Thus if $\mathbf{A}$ is the vector from P to Q and $\mathbf{B}$ is the vector from P to R, then $\mathbf{B}-\mathbf{A}$ is the vector from Q to R (Figure 10.1.13).



Vector difference

Figure 10.1.13

If a trader initially has a commodity vector $\mathbf{A}$ and sells a quantity b_1 of the first commodity and b_2 of the second, his new commodity vector will be the vector difference $\mathbf{A}-\mathbf{B}$.

Given a force vector $\mathbf{F}$, $-\mathbf{F}$ is the force vector of the same magnitude but exactly the opposite direction.

If an object initially has position vector $\mathbf{P}$, then $\mathbf{Q}-\mathbf{P}$ is the displacement vector which will change its position to $\mathbf{Q}$.

THEOREM 2

Let $\mathbf{A}$ *and* $\mathbf{B}$ *be vectors.*

(i) $-\mathbf{0}=\mathbf{0}$,
(ii) $-(-\mathbf{A})=\mathbf{A}$,
(iii) $\mathbf{A}-\mathbf{A}=\mathbf{0}$,
(iv) $\mathbf{B}-\mathbf{A}=\mathbf{B}+(-\mathbf{A})$.

Rule (iv) is illustrated in Figure 10.1.14.

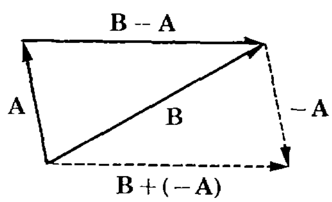


Figure 10.1.14

If $\mathbf{A}$ is a vector with components a_1 and a_2 and c is a scalar, then the *scalar multiple* $c\mathbf{A}$ is the vector with components ca_1, ca_2. Notice that the product of a scalar and a vector is a vector. Geometrically, for positive c, $c\mathbf{A}$ is the vector in the same direction as $\mathbf{A}$ whose length is c times the length of $\mathbf{A}$ (Figure 10.1.15). $(-c)\mathbf{A}$ is the vector in the opposite direction from $\mathbf{A}$ whose length is $c|\mathbf{A}|$. We sometimes write $\mathbf{A}c$ for $c\mathbf{A}$, and $\mathbf{A}/c$ for $(1/c)\mathbf{A}$.

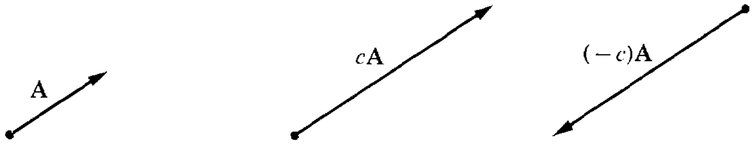


Figure 10.1.15 Scalar Multiples

In physics, *Newton's second law of motion* states that

$$\mathbf{F} = m\mathbf{A}$$

where $\mathbf{F}$ is the force vector acting on an object, $\mathbf{A}$ is the acceleration vector, and the scalar m is the mass of the object.

In economics, if all prices are increased by the same factor c due to inflation, then the new price vector $\mathbf{Q}$ will be a scalar multiple of the initial price vector $\mathbf{P}$,

$$\mathbf{Q} = c\mathbf{P}.$$

THEOREM 3

Let $\mathbf{A}$ *and* $\mathbf{B}$ *be vectors and* s, t *be scalars.*

(i) $0\mathbf{A} = \mathbf{0}, \quad 1\mathbf{A} = \mathbf{A}, \quad (-s)\mathbf{A} = -(s\mathbf{A}).$

(ii) Scalar Associative Law $s(t\mathbf{A}) = (st)\mathbf{A}$.

(iii) Distributive Laws $(s + t)\mathbf{A} = s\mathbf{A} + t\mathbf{A}$,

$$s(\mathbf{A} + \mathbf{B}) = s\mathbf{A} + s\mathbf{B}.$$

(iv) $|s\mathbf{A}| = |s||\mathbf{A}|$.

We shall prove only part (iv) which says that the length of $s\mathbf{A}$ is $|s|$ times the length of $\mathbf{A}$.

Let $\mathbf{A}$ have components a_1, a_2. Then $s\mathbf{A}$ has components sa_1, sa_2.

Thus
$$|s\mathbf{A}| = \sqrt{(sa_1)^2 + (sa_2)^2} = \sqrt{s^2a_1^2 + s^2a_2^2}$$
$$= \sqrt{s^2}\sqrt{a_1^2 + a_2^2} = |s||\mathbf{A}|.$$

A *unit vector* is a vector $\mathbf{U}$ of length one. The two most important unit vectors are the *basis vectors* $\mathbf{i}$ and $\mathbf{j}$. $\mathbf{i}$, the unit vector along the x-axis, has components $(1, 0)$. $\mathbf{j}$, the unit vector along the y-axis, has components $(0, 1)$. Figure 10.1.16 shows $\mathbf{i}$ and $\mathbf{j}$.

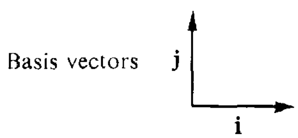


Figure 10.1.16

A vector can be conveniently expressed in terms of the basis vectors.

COROLLARY 1

The vector with components a and b is $a\mathbf{i} + b\mathbf{j}$.

PROOF $a\mathbf{i}$ is the vector from (0, 0) to $(a, 0)$, $b\mathbf{j}$ is the vector from (0, 0) to $(0, b)$. Therefore the sum $a\mathbf{i} + b\mathbf{j}$ is the vector from (0, 0) to (a, b) (Figure 10.1.17).

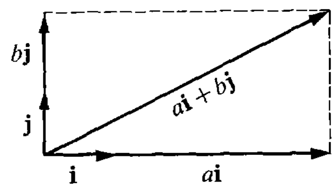


Figure 10.1.17

Sums, differences, scalar multiples, and lengths of vectors can easily be computed using the basis vectors and components. The necessary formulas are given in the next corollary.

COROLLARY 2

Let $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j}$ *and* $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j}$ *be vectors and let c be a scalar.*

(i) $\mathbf{A} + \mathbf{B} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j}$.

(ii) $\mathbf{A} - \mathbf{B} = (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j}$.

(iii) $c\mathbf{A} = (ca_1)\mathbf{i} + (ca_2)\mathbf{j}$.

(iv) $|\mathbf{A}| = \sqrt{a_1^2 + a_2^2}$.

For example, (i) is shown by the computation

$$\begin{aligned}\mathbf{A} + \mathbf{B} &= (a_1\mathbf{i} + a_2\mathbf{j}) + (b_1\mathbf{i} + b_2\mathbf{j}) \\ &= (a_1\mathbf{i} + b_1\mathbf{i}) + (a_2\mathbf{j} + b_2\mathbf{j}) = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j}.\end{aligned}$$

It is illustrated in Figure 10.1.18.

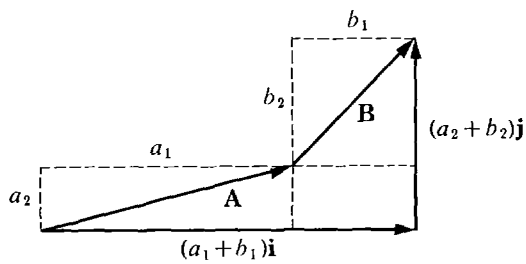


Figure 10.1.18

EXAMPLE 4 Let $\mathbf{A} = 2\mathbf{i} - 5\mathbf{j}$, $\mathbf{B} = \mathbf{i} + 3\mathbf{j}$.

(a) Find $\mathbf{A} + \mathbf{B}$, $\mathbf{A} - \mathbf{B}$, $-\mathbf{A}$, and $6\mathbf{B}$.

$$\mathbf{A}+\mathbf{B}=(2+1)\mathbf{i}+(-5+3)\mathbf{j}=3\mathbf{i}-2\mathbf{j},$$
$$\mathbf{A}-\mathbf{B}=(2-1)\mathbf{i}+(-5-3)\mathbf{j}=\mathbf{i}-8\mathbf{j},$$
$$-\mathbf{A}=(-1)\mathbf{A}=(-1)2\mathbf{i}+(-1)(-5)\mathbf{j}=-2\mathbf{i}+5\mathbf{j},$$
$$6\mathbf{B}=6(\mathbf{i}+3\mathbf{j})=6\mathbf{i}+18\mathbf{j}.$$

(b) Find the vector **D** such that 3**A** + 5**D** = **B**.

$$5\mathbf{D}=-3\mathbf{A}+\mathbf{B},$$
$$\mathbf{D}=\tfrac{1}{5}(-3\mathbf{A}+\mathbf{B}),$$
$$=\tfrac{1}{5}(-3\cdot 2+1)\mathbf{i}+\tfrac{1}{5}(-3(-5)+3)\mathbf{j},$$
$$=-\mathbf{i}+\tfrac{18}{5}\mathbf{j}.$$

EXAMPLE 5 A triangle has vertices (0, 0), (2, −1), and (3, 1) (Figure 10.1.19). Find the vectors counterclockwise around the perimeter of the triangle and check that their sum is the zero vector.

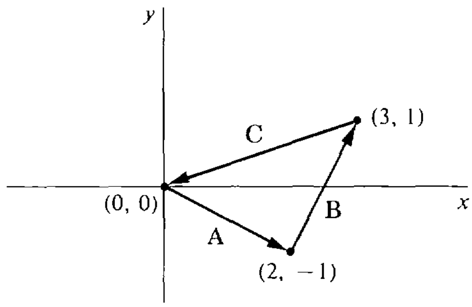


Figure 10.1.19

The three vectors are

$$\mathbf{A}=(2-0)\mathbf{i}+(-1-0)\mathbf{j}=2\mathbf{i}-\mathbf{j},$$
$$\mathbf{B}=(3-2)\mathbf{i}+(1-(-1))\mathbf{j}=\mathbf{i}+2\mathbf{j},$$
$$\mathbf{C}=(0-3)\mathbf{i}+(0-1)\mathbf{j}=-3\mathbf{i}-\mathbf{j}.$$

Their sum is

$$\mathbf{A}+\mathbf{B}+\mathbf{C}=(2+1-3)\mathbf{i}+(-1+2+(-1))\mathbf{j}=0\mathbf{i}+0\mathbf{j}.$$

We need a convenient way of describing the direction as well as the magnitude of a vector. First we define the angle between two vectors (Figure 10.1.20).

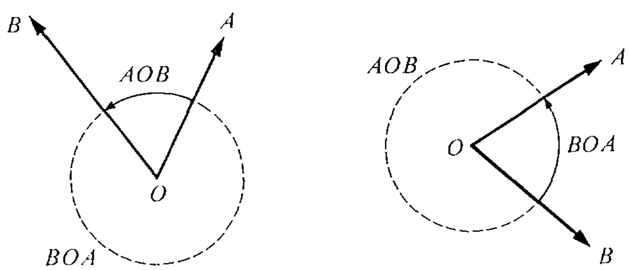


Figure 10.1.20

DEFINITION

Let **A** *and* **B** *be two nonzero vectors in the plane, and let O be the origin. The* ***angle*** *between* **A** *and* **B** *is either the angle AOB or the angle BOA, whichever is in the interval* $[0, \pi]$. *(The angle between* **A** *and the zero vector is undefined.)*

Notice that if AOB is between 0 and π, then BOA is between π and 2π, and vice versa. So exactly one is between 0 and π. The angle θ between **A** and **B** can be computed by using the Law of Cosines from trigonometry, illustrated in Figure 10.1.21.

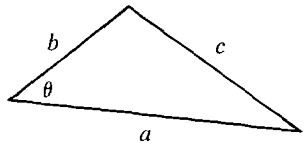


Figure 10.1.21

LAW OF COSINES

In a triangle with sides a, b, c, and angle θ opposite side c,

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

Notice that when $\theta = \pi/2$, $\cos\theta = 0$ and the Law of Cosines reduces to the familiar Theorem of Pythagoras, $c^2 = a^2 + b^2$.

Given vectors **A** and **B** with angle θ between them, we form a triangle with sides $|\mathbf{A}|$, $|\mathbf{B}|$, and $|\mathbf{B} - \mathbf{A}|$. Then by the Law of Cosines,

$$|\mathbf{B} - \mathbf{A}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}|\cos\theta.$$

Solving for $\cos\theta$,

$$\cos\theta = \frac{|\mathbf{A}|^2 + |\mathbf{B}|^2 - |\mathbf{B} - \mathbf{A}|^2}{2|\mathbf{A}||\mathbf{B}|}.$$

Since the arccosine is always between 0 and π,

$$\theta = \arccos\left(\frac{|\mathbf{A}|^2 + |\mathbf{B}|^2 - |\mathbf{B} - \mathbf{A}|^2}{2|\mathbf{A}||\mathbf{B}|}\right).$$

EXAMPLE 6 Find the angle between $\mathbf{A} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{B} = \mathbf{i} + \mathbf{j}$.

$$|\mathbf{A}| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5,$$
$$|\mathbf{B}| = \sqrt{1^2 + 1^2} = \sqrt{2},$$
$$|\mathbf{B} - \mathbf{A}| = \sqrt{(3-1)^2 + (-4-1)^2} = \sqrt{4 + 25} = \sqrt{29},$$
$$\cos\theta = \frac{|\mathbf{A}|^2 + |\mathbf{B}|^2 - |\mathbf{B} - \mathbf{A}|^2}{2|\mathbf{A}||\mathbf{B}|} = \frac{25 + 2 - 29}{2 \cdot 5 \cdot \sqrt{2}}$$
$$= -\frac{2}{10\sqrt{2}} = -\frac{\sqrt{2}}{10}.$$
$$\theta = \arccos\left(-\frac{\sqrt{2}}{10}\right).$$

The direction of a vector can be described in one of three closely related ways: by its direction angles, its direction cosines, or its unit vector.

Let **A** be a nonzero vector. The angles α between **A** and **i**, and β between **A** and **j**, are called the *direction angles* of **A**. The cosines of these angles, $\cos\alpha$ and $\cos\beta$, are called the *direction cosines of* **A**.

The vector $\mathbf{U} = \mathbf{A}/|\mathbf{A}|$ is called the *unit vector* of **A**. **U** has length one, $|\mathbf{U}| = |\mathbf{A}|/|\mathbf{A}| = 1$.

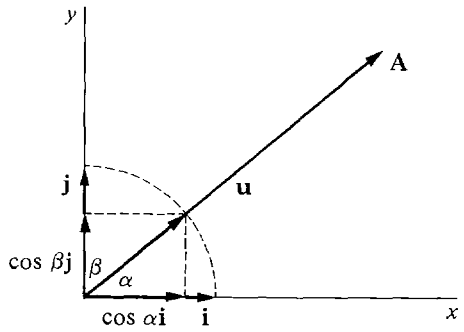


Figure 10.1.22

We can see from Figure 10.1.22 that the components of **U** are the direction cosines of **A**,

$$\mathbf{U} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j}.$$

A vector **A** is determined by its length and its direction cosines,

$$\mathbf{A} = |\mathbf{A}|\,\mathbf{U} = |\mathbf{A}|\cos\alpha\mathbf{i} + |\mathbf{A}|\cos\beta\mathbf{j}.$$

The sum of the squares of the direction cosines is always one, for

$$|\mathbf{U}| = \cos^2\alpha + \cos^2\beta = 1.$$

EXAMPLE 7 Find the unit vector and direction cosines of the given vector.

First find the length, then the unit vector, and then the direction cosines.

(a) $\mathbf{A} = 2\mathbf{i} + \mathbf{j}$ $\quad |\mathbf{A}| = \sqrt{2^2 + 1^2} = \sqrt{5}$

$$\text{Unit vector} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{2\mathbf{i} + \mathbf{j}}{\sqrt{5}}$$

$$\text{Direction cosines} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

(b) $\mathbf{B} = 5\mathbf{i} - 12\mathbf{j}$ $\quad |\mathbf{B}| = \sqrt{5^2 + (-12)^2} = \sqrt{169} = 13$

$$\text{Unit vector} = \frac{\mathbf{B}}{|\mathbf{B}|} = \frac{5\mathbf{i} - 12\mathbf{j}}{13}$$

$$\text{Direction cosines} = (\tfrac{5}{13}, -\tfrac{12}{13}).$$

(c) $\mathbf{C} = \frac{1}{4}\mathbf{j}$ $\quad |\mathbf{C}| = \sqrt{0^2 + (\frac{1}{4})^2} = \frac{1}{4}$

$$\text{Unit vector} = \frac{\frac{1}{4}\mathbf{j}}{\frac{1}{4}} = \mathbf{j}$$

$$\text{Direction cosines} = (0, 1).$$

EXAMPLE 8 Find the vector **A** which has length 6 and direction cosines $(-1/2, \sqrt{3}/2)$.

$$\mathbf{A} = 6(-1/2)\mathbf{i} + 6(\sqrt{3}/2)\mathbf{j} = -3\mathbf{i} + 3\sqrt{3}\mathbf{j}.$$

PROBLEMS FOR SECTION 10.1

In Problems 1–4 find the vector represented by the directed line segment $\overrightarrow{PQ}$.

1 $P = (3, 1),\ \ Q = (4, 3)$

2 $P = (-1, -1),\ \ Q = (2, -2)$

3 $P = (3, 4),\ \ Q = (0, 0)$

4 $P = (0, 0),\ \ Q = (0, 3)$

In Problems 5–8 find the point Q such that **A** is the vector from P to Q.

5 $P = (1, -1),\ \ \mathbf{A} = \mathbf{i} - 3\mathbf{j}$

6 $P = (0, 0),\ \ \mathbf{A} = 3\mathbf{i} - 5\mathbf{j}$

7 $P = (4, 6),\ \ \mathbf{A} = -5\mathbf{i} + 6\mathbf{j}$

8 $P = (3, 3),\ \ \mathbf{A} = 2\mathbf{j}$

In Problems 9–32, find the given vector or scalar, where

$$\mathbf{A} = \mathbf{i} - 2\mathbf{j}, \qquad \mathbf{B} = -4\mathbf{i} + 3\mathbf{j}, \qquad \mathbf{C} = 3\mathbf{i}.$$

9 $\mathbf{A} + \mathbf{B}$

10 $\mathbf{A} + \mathbf{C}$

11 $\mathbf{A} + \mathbf{B} + \mathbf{C}$

12 $-\mathbf{A}$

13 $3\mathbf{A}$

14 $\mathbf{A} - \mathbf{B}$

15 $\mathbf{B} - \mathbf{A}$

16 $3\mathbf{A} + 4\mathbf{B}$

17 $\mathbf{A} - 2\mathbf{B} + 3\mathbf{C}$

18 $|\mathbf{A}|$

19 $|\mathbf{B}|$

20 $|\mathbf{A} + \mathbf{B}|$

21 $|\mathbf{A} - \mathbf{B}|$

22 $|\mathbf{A}| + |\mathbf{B}|$

23 $|6\mathbf{A}|$

24 The vector **D** such that $\mathbf{A} + 2\mathbf{D} = \mathbf{B}$.

25 The vector **D** such that $2\mathbf{A} + 4\mathbf{D} = \mathbf{C} - 3\mathbf{B}$.

26 The unit vector and direction cosines of **A**.

27 The unit vector and direction cosines of **B**.

28 The unit vector and direction cosines of **C**.

29 The angle between **A** and **B**.

30 The angle between **A** and **C**.

31 The angle between **B** and **C**.

32 The angle between $-\mathbf{B}$ and **C**.

33 An object initially has position vector $\mathbf{P} = 3\mathbf{i} + 5\mathbf{j}$ and is displaced by the vector $\mathbf{A} = 4\mathbf{i} - 2\mathbf{j}$. Find its new position vector.

34 An object is displaced first by the vector $\mathbf{A} = -\mathbf{i} - 2\mathbf{j}$ and then by the vector $\mathbf{B} = 4\mathbf{i} - \mathbf{j}$. Find the total displacement vector.

35 Find the displacement vector necessary to change the position vector of an object from $\mathbf{P} = -3\mathbf{i} + 6\mathbf{j}$ to $\mathbf{Q} = 5\mathbf{i} + 4\mathbf{j}$.

36 Three forces are acting on an object, with vectors

$$\mathbf{F} = \mathbf{i} + 3\mathbf{j}, \qquad \mathbf{G} = 2\mathbf{i}, \qquad \mathbf{H} = -2\mathbf{i} - \mathbf{j}.$$

Find the total force on the object.

37 Three forces are acting on an object which is at rest. The first two forces are

$$\mathbf{F}_1 = -6\mathbf{i} + 9\mathbf{j}, \qquad \mathbf{F}_2 = 10\mathbf{i} - 3\mathbf{j}.$$

Find the third force $\mathbf{F}_3$.

38 An object of mass 10 is being accelerated so that its acceleration vector is $\mathbf{A} = 5\mathbf{i} - 6\mathbf{j}$. Find the total force acting on the object.

39 An object is displaced by the vector $3\mathbf{i} - 4\mathbf{j}$. Find the distance it is moved.

40 An object has the velocity vector $\mathbf{V} = \mathbf{i} - \mathbf{j}$. Find its speed.

41 A trader initially has the commodity vector $\mathbf{A} = 3\mathbf{i} + \mathbf{j}$ and buys the commodity vector $\mathbf{B} = \mathbf{i} + 2\mathbf{j}$. Find his new total commodity vector.

42 Two traders initially have commodity vectors $\mathbf{A}_0 = 4\mathbf{i} + \mathbf{j}$, $\mathbf{B}_0 = 3\mathbf{i} + 6\mathbf{j}$. After trading with each other, trader A has the commodity vector $\mathbf{A}_1 = 3\mathbf{i} + 3\mathbf{j}$. Find the new commodity vector $\mathbf{B}_1$ of trader B.

43 A trader initially has the commodity vector $\mathbf{A} = 15\mathbf{i} + 12\mathbf{j}$ and sells the commodity vector $5\mathbf{i} + 10\mathbf{j}$. Find his new commodity vector.

44 A pair of commodities initially has the price vector $\mathbf{P} = 6\mathbf{i} + 9\mathbf{j}$. Due to inflation all prices are increased by 10%. Find the new price vector.

45 Find the vector with length 4 and direction cosines $(-\sqrt{2}/2, \sqrt{2}/2)$.

46 Find the vector with length 4 and direction cosines $(-1, 0)$.

47 Find the vector with length 10 and direction cosines $(\frac{3}{5}, \frac{4}{5})$.

In Problems 48–50 find the vectors counterclockwise around the perimeter of the polygon with the given vertices.

48 $(0, 0), (1, 0), (0, 1)$.

49 $(1, 1), (3, 0), (5, 2), (0, 4)$.

50 The regular hexagon inscribed in the unit circle $x^2 + y^2 = 1$ with the initial vertex $(1, 0)$.

51 Use the Triangle Inequality to prove the following.

$$|\mathbf{A} - \mathbf{B}| \le |\mathbf{A}| + |\mathbf{B}|,$$
$$|\mathbf{A}| - |\mathbf{B}| \le |\mathbf{A} + \mathbf{B}|,$$
$$|\mathbf{A} + \mathbf{B} + \mathbf{C}| \le |\mathbf{A}| + |\mathbf{B}| + |\mathbf{C}|.$$

□ **52** Prove that for every nonzero vector $\mathbf{A}$ and positive scalar s, there are exactly two scalar multiples $t\mathbf{A}$ of length s.

□ **53** Prove that two nonzero vectors $\mathbf{A}$ and $\mathbf{B}$ have the same direction cosines if and only if $\mathbf{B} = t\mathbf{A}$ for some positive t.

□ **54** Prove the Commutative Law for vector addition.

□ **55** Prove the Distributive Laws for scalar multiples.

□ **56** Prove the Triangle Inequality. *Hint*: Assume

$$\sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2} > \sqrt{a_1^2 + a_2^2} + \sqrt{b_1^2 + b_2^2}$$

and get a contradiction. This is done by squaring both sides, simplifying, and then squaring and simplifying again.

10.2 VECTORS AND PLANE GEOMETRY

In this section we apply the algebra of two-dimensional vectors to plane geometry.

Given a point $P(p_1, p_2)$ in the plane, the *position vector* of P is the vector $\mathbf{P}$ from the origin to P (Figure 10.2.1). $\mathbf{P}$ has components p_1 and p_2, so

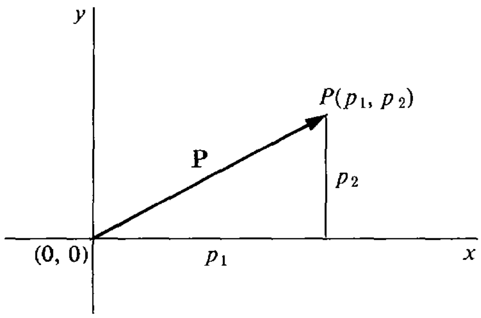


Figure 10.2.1 The position vector

$$\mathbf{P} = p_1\mathbf{i} + p_2\mathbf{j}.$$

If A and B are two points in the plane with position vectors $\mathbf{A}$ and $\mathbf{B}$, then the vector from A to B is the vector difference $\mathbf{B} - \mathbf{A}$. This can be seen from Figure 10.2.2.

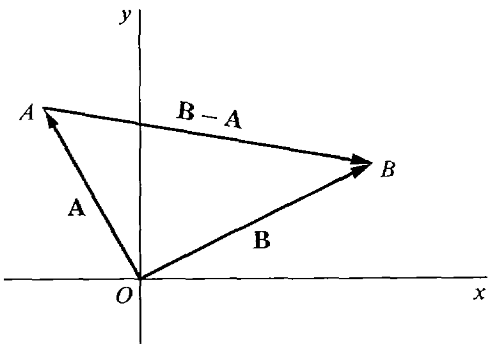


Figure 10.2.2

In Section 1.3, we saw that a line in the plane may be defined as the graph of a linear equation

$$ax + by = c$$

where a and b are not both zero (Figure 10.2.3). We shall call the above equation a *scalar equation* of the line.

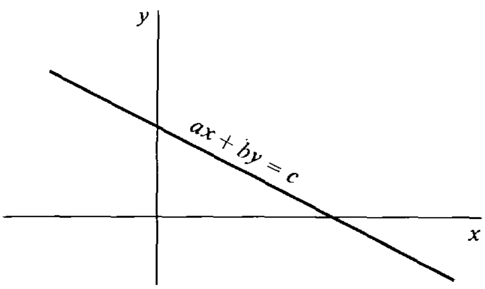


Figure 10.2.3

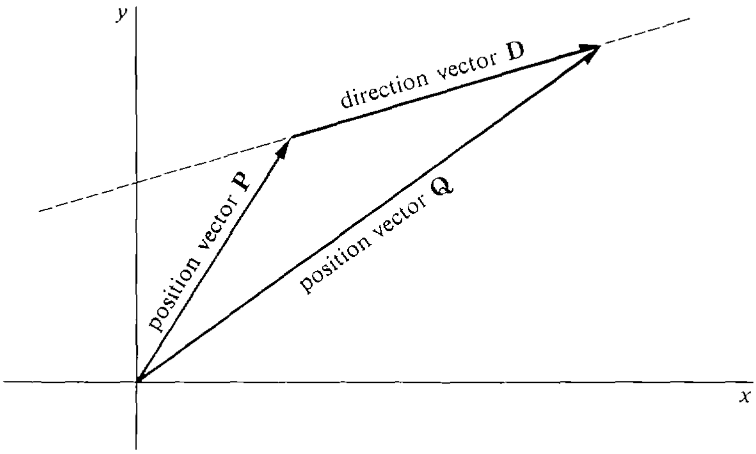


Figure 10.2.4

The position vector of any point P on a line L is called a *position vector* of L. If P and Q are two distinct points on L, the vector $\mathbf{D}$ from P to Q is called a *direction vector* of L. Thus $\mathbf{D} = \mathbf{Q} - \mathbf{P}$ (Figure 10.2.4).

Theorem 1 will show how to represent a line by a *vector equation*. Let us use the symbol X for the *variable point* $X(x, y)$, and the symbol $\mathbf{X}$ for the *variable vector* $\mathbf{X} = x\mathbf{i} + y\mathbf{j}$. (see Figure 10.2.5).

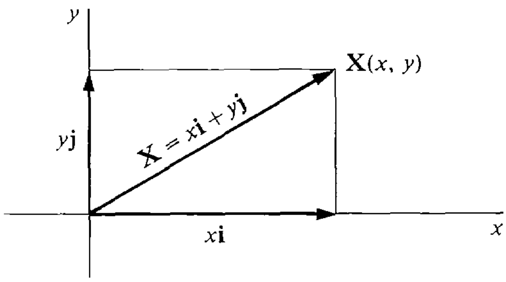


Figure 10.2.5

THEOREM 1

A line L is uniquely determined by a position vector $\mathbf{P}$ and a direction vector $\mathbf{D}$. L has the scalar equation

$$xd_2 - yd_1 = p_1d_2 - p_2d_1$$

and the vector equation

$$\mathbf{X} = \mathbf{P} + t\mathbf{D}.$$

The vector equation means that L is the set of all points X such that $\mathbf{X} = \mathbf{P} + t\mathbf{D}$ for some t.

PROOF Let L be any line with position vector $\mathbf{P}$ and direction vector $\mathbf{D}$. We must show that:

(i) L has the scalar equation given in Theorem 1.
(ii) If $\mathbf{X} = \mathbf{P} + t\mathbf{D}$ for some t then X is a point of L.
(iii) If X is a point of L then $\mathbf{X} = \mathbf{P} + t\mathbf{D}$ for some t.

(i) $\mathbf{D}$ is the vector from A to B where A and B are points on L. Since L is the line through A and B, it has the scalar equation

$$(x - a_1)(b_2 - a_2) = (y - a_2)(b_1 - a_1),$$
$$(x - a_1)d_2 = (y - a_2)d_1,$$
$$xd_2 - yd_1 = a_1d_2 - a_2d_1.$$

This equation holds for the point P of L,

$$p_1d_2 - p_2d_1 = a_1d_2 - a_2d_1.$$

Combining the last two equations we get the required equation:

(1) $$xd_2 - yd_1 = p_1d_2 - p_2d_1.$$

(ii) Let X be a point such that $\mathbf{X} = \mathbf{P} + t\mathbf{D}$ for some t. Then

$$x = p_1 + td_1, \qquad y = p_2 + td_2,$$
$$d_2x - d_1y = d_2p_1 + d_2td_1 - d_1p_2 - d_1td_2 = d_2p_1 - d_1p_2,$$

so X is a point of L.

(iii) Let X be a point of L. If $d_1 \neq 0$ we set $t = (x - p_1)/d_1$, and using Equation 1 we get $\mathbf{X} = \mathbf{P} + t\mathbf{D}$. The case $d_2 \neq 0$ is similar. Therefore $\mathbf{X} = \mathbf{P} + t\mathbf{D}$ is a vector equation for L (Figure 10.2.6).

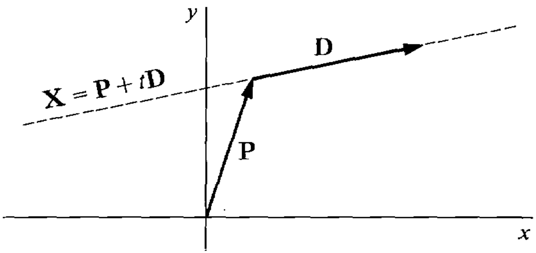


Figure 10.2.6

The line with position vector **P** and direction vector **D**

The vector equation $\mathbf{X} = \mathbf{P} + t\mathbf{D}$ can be put in the form

$$x\mathbf{i} + y\mathbf{j} = (p_1 + td_1)\mathbf{i} + (p_2 + td_2)\mathbf{j}.$$

It can also be written as a pair of *parametric equations*

$$x = p_1 + td_1, \qquad y = p_2 + td_2.$$

EXAMPLE 1 Find a vector equation for the line through the two points $A(2, 1)$ and $B(-4, 0)$, shown in Figure 10.2.7.

The vector $\mathbf{D} = \mathbf{B} - \mathbf{A}$ from A to B is given by

$$\mathbf{D} = (-4 - 2)\mathbf{i} + (0 - 1)\mathbf{j} = -6\mathbf{i} - \mathbf{j}.$$

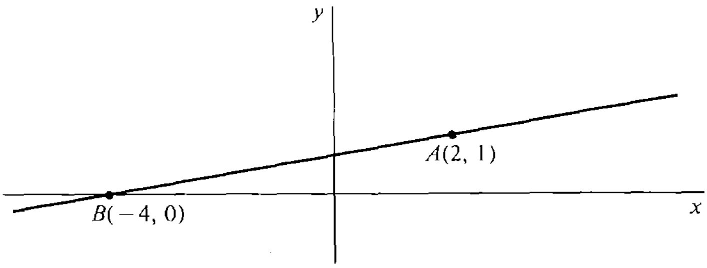


Figure 10.2.7

Since **A** is a position vector and **D** a direction vector of the line, the line has the vector equation

$$\mathbf{X} = \mathbf{A} + t\mathbf{D}$$
$$= 2\mathbf{i} + \mathbf{j} + t(-6\mathbf{i} - \mathbf{j}).$$

In general, the line L through points A and B has the vector equation $\mathbf{X} = \mathbf{A} + t(\mathbf{B} - \mathbf{A})$ because **A** is a position vector and $\mathbf{B} - \mathbf{A}$ is a direction vector of L.

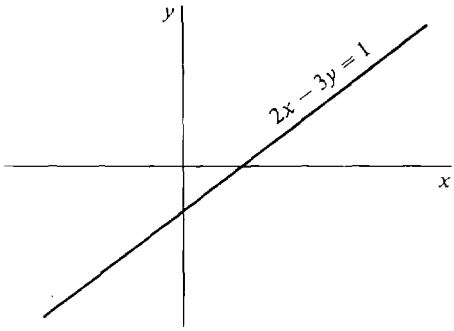


Figure 10.2.8

EXAMPLE 2 Find a vector equation for the line in Figure 10.2.8:

$$2x - 3y = 1.$$

Step 1 Find two points on the line by taking two values of x and solving for y.

$$x = 0, \quad 0 - 3y = 1, \quad y = -\tfrac{1}{3}, \quad (0, -\tfrac{1}{3}).$$
$$x = 1, \quad 2 - 3y = 1, \quad y = \tfrac{1}{3}, \quad (1, \tfrac{1}{3}).$$

Step 2 Find a position and direction vector.

$$\mathbf{P} = 0\mathbf{i} + (-\tfrac{1}{3})\mathbf{j} = -\tfrac{1}{3}\mathbf{j}.$$
$$\mathbf{D} = (1 - 0)\mathbf{i} + (\tfrac{1}{3} - (-\tfrac{1}{3}))\mathbf{j} = \mathbf{i} + \tfrac{2}{3}\mathbf{j}.$$

Step 3 Use Theorem 1. The vector equation is

$$\mathbf{X} = \mathbf{P} + t\mathbf{D}$$
$$= -\tfrac{1}{3}\mathbf{j} + t(\mathbf{i} + \tfrac{2}{3}\mathbf{j}).$$

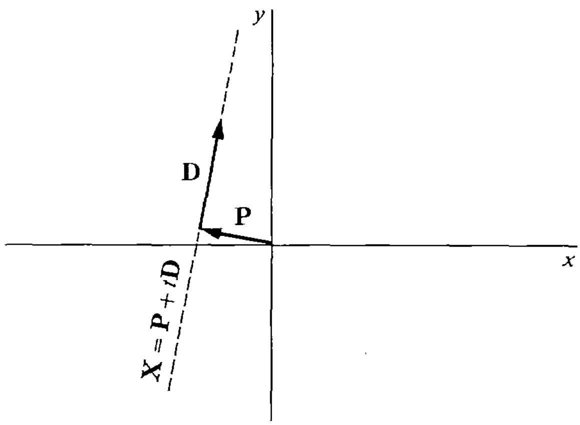


Figure 10.2.9

EXAMPLE 3 Find a scalar equation for the line in Figure 10.2.9:

$$\mathbf{X} = -4\mathbf{i} + \mathbf{j} + t(\mathbf{i} + 6\mathbf{j}).$$

First method By Theorem 1, the line has the equation

$$xd_2 - yd_1 = p_1d_2 - p_2d_1,$$
$$6x - y = (-4)\cdot 6 - 1\cdot 1,$$
$$6x - y = -25.$$

Second method We convert the vector equation to parametric equations and then eliminate t.

$$x = -4 + t, \qquad y = 1 + 6t,$$
$$t = x + 4, \qquad y = 1 + 6(x + 4),$$
$$y = 25 + 6x.$$

This is equivalent to the first solution.

EXAMPLE 4 Determine whether the three points

$$A(1, 3), \qquad B(2, 5), \qquad C(3, 10)$$

are on the same line.

The line L through A and B has the vector equation

$$\mathbf{X} = \mathbf{A} + t(\mathbf{B} - \mathbf{A})$$
$$= \mathbf{i} + 3\mathbf{j} + t(\mathbf{i} + 2\mathbf{j}) = (1 + t)\mathbf{i} + (3 + 2t)\mathbf{j}.$$

The only point on L with x component 3 is given by

$$3 = 1 + t, \qquad t = 2, \qquad \mathbf{P} = 3\mathbf{i} + 7\mathbf{j}.$$

Since C is another point with x component 3, C is not on L. Therefore A, B, and C are not on the same line, as we see in Figure 10.2.10.

Some applications of vectors to plane geometry follow.

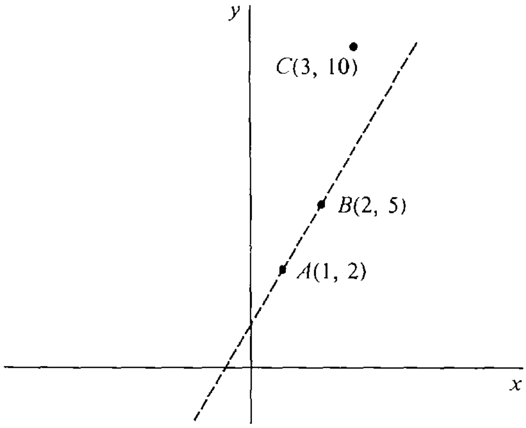


Figure 10.2.10

EXAMPLE 5 Let A and B be two distinct points. Prove that the midpoint of the line segment AB is the point P with position vector $\mathbf{P} = \frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{B}$.

PROOF We shall prove that the point P is on the line AB and is equidistant from A and B (see Figure 10.2.11). The line through A and B has the direction

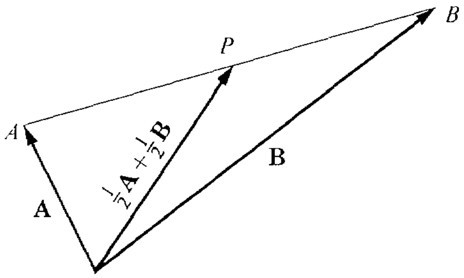


Figure 10.2.11

vector $\mathbf{D} = \mathbf{B} - \mathbf{A}$. The vector $\mathbf{P}$ has the form

$$\mathbf{P} = \tfrac{1}{2}\mathbf{A} + \tfrac{1}{2}\mathbf{B} = \mathbf{A} + \tfrac{1}{2}(\mathbf{B} - \mathbf{A}) = \mathbf{A} + \tfrac{1}{2}\mathbf{D}.$$

Therefore by Theorem 1, P is on the line AB. To prove that P is equidistant, we show that the vector from A to P is the same as the vector from P to B

$$\mathbf{P} - \mathbf{A} = \tfrac{1}{2}\mathbf{A} + \tfrac{1}{2}\mathbf{B} - \mathbf{A} = \tfrac{1}{2}\mathbf{B} - \tfrac{1}{2}\mathbf{A}.$$
$$\mathbf{B} - \mathbf{P} = \mathbf{B} - \tfrac{1}{2}\mathbf{A} - \tfrac{1}{2}\mathbf{B} = \tfrac{1}{2}\mathbf{B} - \tfrac{1}{2}\mathbf{A}.$$

EXAMPLE 6 Find the midpoint of the line segment from $A(-1, 2)$ to $B(3, 3)$ (Figure 10.2.12).

The points have position vectors

$$\mathbf{A} = -\mathbf{i} + 2\mathbf{j}, \qquad \mathbf{B} = 3\mathbf{i} + 3\mathbf{j}.$$

The midpoint $\mathbf{P}$ has the position vector

$$\mathbf{P} = \tfrac{1}{2}\mathbf{A} + \tfrac{1}{2}\mathbf{B} = \tfrac{1}{2}(-\mathbf{i} + 2\mathbf{j}) + \tfrac{1}{2}(3\mathbf{i} + 3\mathbf{j}) = \mathbf{i} + \tfrac{5}{2}\mathbf{j}.$$

Therefore P is the point $(1, \frac{5}{2})$.

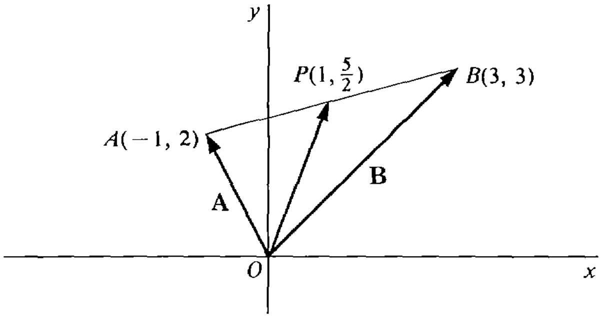


Figure 10.2.12

A four-sided figure whose opposite sides represent equal vectors is called a *parallelogram.*

EXAMPLE 7 Prove that the diagonals of a parallelogram bisect each other.

PROOF We are given a parallelogram $ABCD$, shown in Figure 10.2.13.

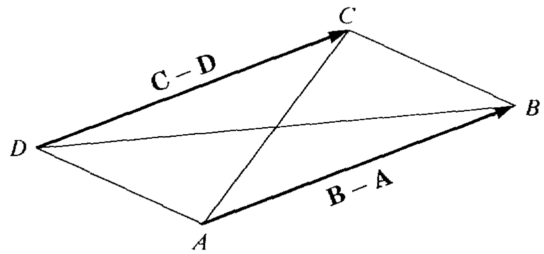


Figure 10.2.13

Since the opposite sides represent equal vectors, we have

(2) $$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D}.$$

The diagonal AC has midpoint $\frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{C}$ and the other diagonal BD has midpoint $\frac{1}{2}\mathbf{B} + \frac{1}{2}\mathbf{D}$. We show that these two midpoints are equal. The Equation 2 gives

$$\mathbf{C} = \mathbf{B} - \mathbf{A} + \mathbf{D}.$$

Then $$\tfrac{1}{2}\mathbf{A} + \tfrac{1}{2}\mathbf{C} = \tfrac{1}{2}\mathbf{A} + \tfrac{1}{2}(\mathbf{B} - \mathbf{A} + \mathbf{D}) = \tfrac{1}{2}\mathbf{B} + \tfrac{1}{2}\mathbf{D}.$$

Thus the two diagonals meet at their midpoints.

EXAMPLE 8 Prove that the lines from the vertices of a triangle ABC to the midpoints of the opposite sides all meet at the single point P given by

$$\mathbf{P} = \tfrac{1}{3}\mathbf{A} + \tfrac{1}{3}\mathbf{B} + \tfrac{1}{3}\mathbf{C}.$$

PROOF We are given triangle ABC, shown in Figure 10.2.14. Let A', B', C' be the midpoints of the opposite sides. We prove that all three lines AA', BB', CC' pass through the point P.

The point A' has position vector

$$\mathbf{A}' = \tfrac{1}{2}\mathbf{B} + \tfrac{1}{2}\mathbf{C}.$$

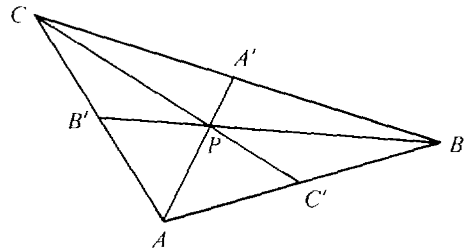


Figure 10.2.14

The line AA' has the direction vector $\mathbf{A}' - \mathbf{A}$. AA' has the vector equation

$$\mathbf{X} = \mathbf{A} + t(\mathbf{A}' - \mathbf{A}).$$

The computation below shows that P is on the line AA'

$$\mathbf{P} = \tfrac{1}{3}\mathbf{A} + (\tfrac{1}{3}\mathbf{B} + \tfrac{1}{3}\mathbf{C}) = \tfrac{1}{3}\mathbf{A} + \tfrac{2}{3}\mathbf{A}' = \mathbf{A} + \tfrac{2}{3}(\mathbf{A}' - \mathbf{A}).$$

A similar proof shows that P is on BB' and CC'.

PROBLEMS FOR SECTION 10.2

In Problems 1–14, find a vector equation for the given line.

1 The line through $P(3, -1)$ with direction vector $\mathbf{D} = -\mathbf{i} + \mathbf{j}$.

2 The line through $P(0, 0)$ with direction vector $\mathbf{D} = \mathbf{i} + 2\mathbf{j}$.

3 The line with parametric equations $x = 3 - 2t$, $y = 4 + 5t$.

4 The line with parametric equations $x = 4t$, $y = 1 + t$.

5 The line through the points $P(1, 4)$ and $Q(2, -1)$.

6 The line through the points $P(5, 5)$ and $Q(-6, 6)$.

7 The vertical line through $P(2, 5)$.

8 The horizontal line through $P(4, 1)$.

9 The line $y = 2 + 5x$.

10 The line $x + y = 3$.

11 The line $y = 3$.

12 The line $x = y$.

13 The line through $P(6, 5)$ with slope -3.

14 The line through $P(1, 2)$ with slope 4.

15 Find a scalar equation for the line $\mathbf{X} = 3\mathbf{i} - 4\mathbf{j} + t(\mathbf{i} - 2\mathbf{j})$.

16 Find a scalar equation for the line $\mathbf{X} = 2\mathbf{i} + t(-\mathbf{i} + 4\mathbf{j})$.

17 Find a scalar equation for the line $\mathbf{X} = \mathbf{i} + 3\mathbf{j} + 4t\mathbf{i}$.

18 Find a scalar equation for the line with parametric equations $x = 3 - 4t$, $y = 1 + 2t$.

In Problems 19–24, determine whether the given three points are on a line.

19 $A(1, 1)$, $B(2, 4)$, $C(-1, -2)$.

20 $A(1, 3)$, $B(2, 5)$, $C(-1, -1)$.

21 $A(4, 0)$, $B(0, 1)$, $C(12, -2)$.

22 $A(6, 3)$, $B(5, 7)$, $C(4, 10)$.

23 $A(5, -1)$, $B(5, 2)$, $C(5, 6)$.

24 $A(-3, 2)$, $B(-3, 3)$, $C(0, 0)$.

25 Find the midpoint of the line AB where $A = (2, 5)$, $B = (-6, 1)$.

26 Find the midpoint of the line AB where $A = (-1, -4)$, $B = (9, 16)$.

27 Find the midpoint of the line AB where $A = (5, 10)$, $B = (-1, 10)$.

28 Find the point of intersection of the diagonals of the parallelogram $A(1, 4)$, $B(6, 4)$, $C(6, 6)$, $D(1, 6)$.

29 Find the point of intersection of the diagonals of the parallelogram $A(2, 0)$, $B(5, 1)$, $C(6, 6)$, $D(3, 5)$.

30 Find the point of intersection of the lines from the vertices to the midpoints of the opposite sides of the triangle ABC, where $A = (1, 4)$, $B = (2, -1)$, $C = (6, 3)$.

31 Prove that the slope of a line with direction vector $\mathbf{D} = d_1\mathbf{i} + d_2\mathbf{j}$ is $m = d_2/d_1$ (vertical if $d_1 = 0$).

□ 32 Prove that if the diagonals of a four-sided figure bisect each other then the figure is a parallelogram. (Converse of Example 7.)

□ 33 Prove that if the opposite sides of a four-sided figure are scalar multiples of each other then the figure is a parallelogram (i.e., the opposite sides are equal as vectors).

□ 34 Let ABC be a triangle and let A_1, B_1, C_1 be the midpoints of the sides opposite A, B, C respectively. Show that the line AA_1 bisects the line B_1C_1.

□ 35 Show that the midpoints of the sides of any four-sided figure are the vertices of a parallelogram.

□ 36 Given a triangle ABC, let D be the midpoint of AB and E the midpoint of AC. Show that DE is parallel to BC and DE has half the length of BC. *Hint*: Show that $\mathbf{E} - \mathbf{D} = \frac{1}{2}(\mathbf{C} - \mathbf{B})$.

10.3 VECTORS AND LINES IN SPACE

In the preceding section, we used the algebra of vectors to prove some facts from plane geometry. This approach really comes into its own in solid geometry. Without using vectors, it is quite hard to define such basic concepts as a straight line, or the angle between two lines, in space. In this section we shall develop geometry in three-dimensional space with vectors as our starting point. The notions of a straight line and an angle in space will be defined using vectors, and we shall use vector algebra to solve problems about lines and angles. Later on in this chapter we shall continue our development of solid geometry, using vectors to study planes in space.

Vectors in space are developed in the same way as vectors in the plane. Three-dimensional space has three perpendicular coordinate axes, x, y, and z, as shown in Figure 10.3.1. This is called a *right-handed coordinate system*, because the

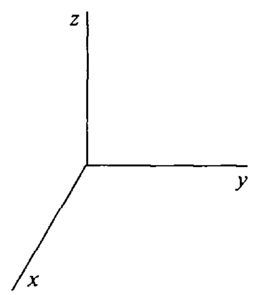


Figure 10.3.1

right thumb, forefinger, and middle finger can point in the direction of the positive x, y, and z axes respectively.

A point in space has three coordinates, one along each axis. We thus identify a point in space with an ordered triple of real numbers, as in Figure 10.3.2.

Given two points $P(p_1, p_2, p_3)$ and $Q(q_1, q_2, q_3)$ in space, the directed line segment PQ has the x-component $q_1 - p_1$, y-component $q_2 - p_2$, and z-component $q_3 - p_3$ (Figure 10.3.3).

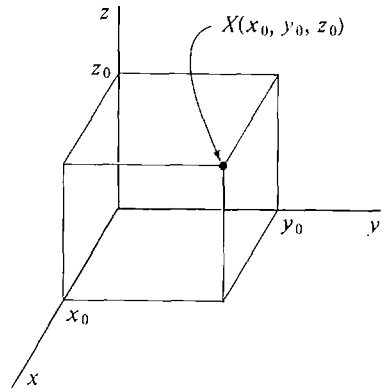


Figure 10.3.2

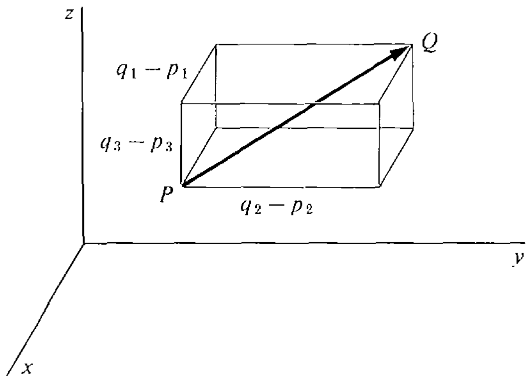


Figure 10.3.3 A Directed Line Segment

The family of all directed line segments in space which have the same three components as $\overrightarrow{PQ}$ is called the *vector in three dimensions*, or the *vector in space*, represented by $\overrightarrow{PQ}$.

The examples of vectors which we discussed in the plane also arise naturally in space. In space, position, velocity, acceleration, force, and displacement are vector quantities with three dimensions. In an economic model with three commodities, the commodity and price vectors have three dimensions.

Vectors in n dimensions arise quite naturally in economics, as commodity and price vectors in an economic model with n commodities. They also arise in more advanced parts of physics, such as quantum mechanics.

Sums, negatives, differences, and scalar multiples of vectors in three dimensions are defined exactly as in two dimensions. The *length*, or *norm*, of a vector **A** with components a_1, a_2, a_3 is defined by

$$|\mathbf{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

THEOREM 1

All the rules for vector algebra given in Section 10.1 *hold for vectors in three dimensions.*

These rules are in Theorems 1, 2, and 3 of Section 10.1, and include the Triangle Inequality.

In space, there are three *basis vectors*, denoted by **i**, **j**, and **k**. **i** has components (1, 0, 0), **j** has components (0, 1, 0), and **k** has components (0, 0, 1). **i**, **j**, and **k** are shown in Figure 10.3.4. As in the case of two dimensions, we see that

$a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ *is the vector in three dimensions with components* a, b, *and* c.

The rules for computing vectors by their components take the following form in three dimensions.

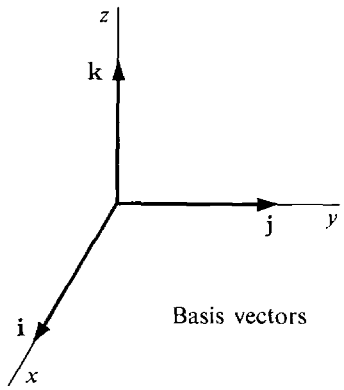


Figure 10.3.4

COROLLARY 1 (Three Dimensions)

Let $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ *and* $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ *be vectors in three dimensions and let* c *be a scalar.*

(i) $\mathbf{A} + \mathbf{B} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}$.

(ii) $\mathbf{A} - \mathbf{B} = (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j} + (a_3 - b_3)\mathbf{k}$.

(iii) $c\mathbf{A} = (ca_1)\mathbf{i} + (ca_2)\mathbf{j} + (ca_3)\mathbf{k}$.

EXAMPLE 1 Given $\mathbf{A} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{B} = 2\mathbf{i} - 2\mathbf{k}$, find $\mathbf{A} + \mathbf{B}$, $\mathbf{A} - \mathbf{B}$, $|\mathbf{A}|$, and $3\mathbf{A}$.

$$\mathbf{A} + \mathbf{B} = (1 + 2)\mathbf{i} + (-1 + 0)\mathbf{j} + (2 - 2)\mathbf{k} = 3\mathbf{i} - \mathbf{j}.$$

$$\mathbf{A} - \mathbf{B} = (1 - 2)\mathbf{i} + (-1 - 0)\mathbf{j} + (2 - (-2))\mathbf{k} = -\mathbf{i} - \mathbf{j} + 4\mathbf{k}.$$

$$|\mathbf{A}| = \sqrt{1^2 + (-1)^2 + (2^2)} = \sqrt{6}.$$

$$3\mathbf{A} = 3\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}.$$

The Law of Cosines gives us a formula for the angle between two vectors in space. In fact, we shall use the Law of Cosines to *define* the angle between two vectors.

DEFINITION

Let **A** *and* **B** *be two nonzero vectors in space. The* ***angle*** *between* **A** *and* **B** *is the angle* θ *between* 0 *and* π *such that*

$$\cos\theta = \frac{|\mathbf{A}|^2 + |\mathbf{B}|^2 - |\mathbf{B} - \mathbf{A}|^2}{2|\mathbf{A}||\mathbf{B}|}.$$

One can prove from the Triangle Inequality that the above quantity is always between -1 and 1, and therefore is the cosine of some angle θ (Problem 42 at the end of this section).

EXAMPLE 2 Find the angle between $\mathbf{A} = \mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{B} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$.

$$|\mathbf{A}| = \sqrt{1^2 + (-1)^2 + (-1)^2} = \sqrt{3}.$$

$$|\mathbf{B}| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}.$$

$$|\mathbf{B} - \mathbf{A}| = \sqrt{(2-1)^2 + (1-(-1))^2 + (1-(-1))^2}$$

$$= \sqrt{1^2 + 2^2 + 2^2} = 3.$$

$$\cos\theta = \frac{3 + 6 - 9}{2\sqrt{3}\sqrt{6}} = 0. \qquad \theta = \arccos 0 = \frac{\pi}{2}.$$

The *direction angles* of a nonzero vector **A** in space are the three angles α, β, γ between **A** and **i**, **j**, **k** respectively. The cosines of the direction angles are called the *direction cosines* of **A**. Let us compute the direction cosines in terms of the components of **A**.

$$\cos\alpha = \frac{|\mathbf{A}|^2 + |\mathbf{i}|^2 - |\mathbf{i} - \mathbf{A}|^2}{2|\mathbf{A}||\mathbf{i}|}$$

$$= \frac{a_1^2 + a_2^2 + a_3^2 + 1 - ((1 - a_1)^2 + a_2^2 + a_3^2)}{2|\mathbf{A}|}$$

$$= \frac{a_1^2 + a_2^2 + a_3^2 + 1 - 1 + 2a_1 - a_1^2 - a_2^2 - a_3^2}{2|\mathbf{A}|}$$

$$= \frac{a_1}{|\mathbf{A}|}.$$

The computations for β and γ are similar. Thus

$$\cos\alpha = \frac{a_1}{|\mathbf{A}|}, \qquad \cos\beta = \frac{a_2}{|\mathbf{A}|}, \qquad \cos\gamma = \frac{a_3}{|\mathbf{A}|}.$$

The *unit vector* of **A** is defined as

$$\mathbf{U} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{a_1}{|\mathbf{A}|}\mathbf{i} + \frac{a_2}{|\mathbf{A}|}\mathbf{j} + \frac{a_3}{|\mathbf{A}|}\mathbf{k}.$$

As in the two-dimensional case, the components of **U**, shown in Figure 10.3.5, are

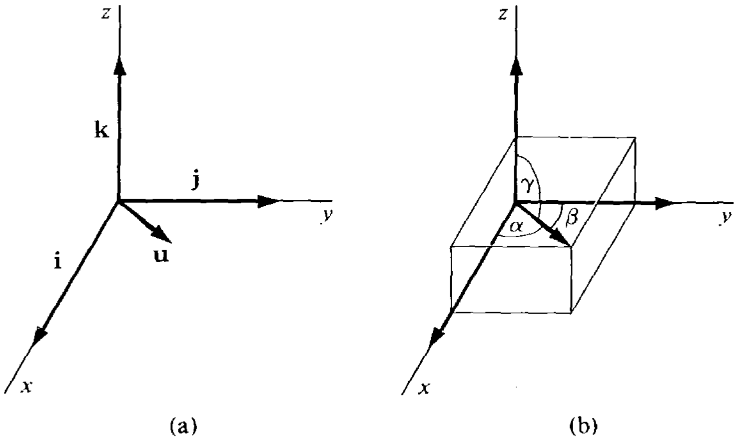


Figure 10.3.5

the direction cosines of **A**,

$$\mathbf{U} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}.$$

Again, **A** is determined by its length and direction cosines, and the sum of the squares of the direction cosines is one,

$$\mathbf{A} = |\mathbf{A}|\cos\alpha\mathbf{i} + |\mathbf{A}|\cos\beta\mathbf{j} + |\mathbf{A}|\cos\gamma\mathbf{k},$$
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

EXAMPLE 3 Find the unit vectors and direction cosines of the vector $\mathbf{A} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

We first find the length, then the unit vector, then the direction cosines.

$$|\mathbf{A}| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3.$$
$$\mathbf{U} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{2\mathbf{i} + \mathbf{j} - 2\mathbf{k}}{3}.$$

Direction cosines $= (\frac{2}{3}, \frac{1}{3}, -\frac{2}{3})$.

The *position vector* of a point $P(p_1, p_2, p_3)$ in space is the vector

$$\mathbf{P} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}.$$

X denotes the *variable vector*

$$\mathbf{X} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

We shall now define the notion of a line in space. The simplest way to describe a line in space is by a vector equation.

DEFINITION

Let **P** *be a vector and* **D** *a nonzero vector in space. The* ***line*** *with the vector equation* $\mathbf{X} = \mathbf{P} + t\mathbf{D}$ *is the set of all points X such that* $\mathbf{X} = \mathbf{P} + t\mathbf{D}$ *for some scalar t.*

The vector equation can also be written as a set of parametric equations

$$x = p_1 + d_1 t, \qquad y = p_2 + d_2 t, \qquad z = p_3 + d_3 t.$$

If t is time, the line is the path of a moving particle in space given by these parametric equations.

The three coordinate axes are lines with the following vector equations.

$$x\text{-axis}: \quad \mathbf{X} = t\mathbf{i},$$
$$y\text{-axis}: \quad \mathbf{X} = t\mathbf{j},$$
$$z\text{-axis}: \quad \mathbf{X} = t\mathbf{k}.$$

EXAMPLE 4 Find a vector equation for the line L with the parametric equations

$$x = 3t + 2, \qquad y = 0t - 4, \qquad z = t + 0.$$

Let $$\mathbf{P} = 2\mathbf{i} - 4\mathbf{j}, \qquad \mathbf{D} = 3\mathbf{i} + \mathbf{k},$$

then L has the vector equation

$$\mathbf{X} = \mathbf{P} + t\mathbf{D}, \quad \text{or} \quad \mathbf{X} = (2\mathbf{i} - 4\mathbf{j}) + t(3\mathbf{i} + \mathbf{k}).$$

L is shown in Figure 10.3.6.

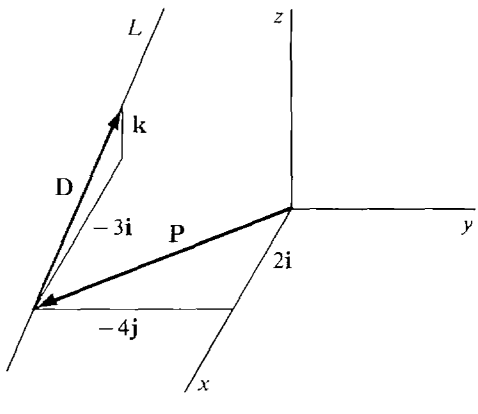


Figure 10.3.6

If A is a point on L, let us call $\mathbf{A}$ a *position vector* of L. A vector $\mathbf{D}$ is said to be a *direction vector* of L if $\mathbf{D}$ is the vector from one point of L to another point of L. Thus if $\mathbf{A}$ and $\mathbf{B}$ are distinct position vectors of L, then $\mathbf{B} - \mathbf{A}$ is a direction vector of L (Figure 10.3.7).

The next theorem shows that a line in space is uniquely determined by a position vector and a direction vector. That is, if two lines L and M have a position vector and direction vector in common, then L and M must be the same line.

THEOREM 2

Given a vector $\mathbf{P}$ *and a nonzero vector* $\mathbf{D}$, *the line* $\mathbf{X} = \mathbf{P} + t\mathbf{D}$ *is the unique line with position vector* $\mathbf{P}$ *and direction vector* $\mathbf{D}$.

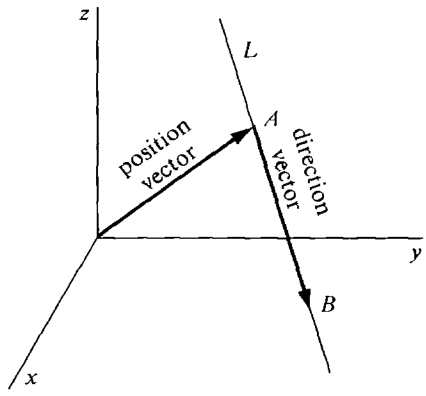


Figure 10.3.7

PROOF Let L be the line $\mathbf{X} = \mathbf{P} + t\mathbf{D}$. Setting $t = 0$ and $t = 1$ we see that $\mathbf{P}$ and $\mathbf{P} + \mathbf{D}$ are position vectors of L, so $\mathbf{D}$ is a direction vector of L.

Let $\mathbf{X} = \mathbf{Q} + s\mathbf{E}$ be any line M with position vector $\mathbf{P}$ and direction vector $\mathbf{D}$. We show $\mathbf{X} = \mathbf{Q} + s\mathbf{E}$ is another vector equation for L. For some s_0,

$$\mathbf{P} = \mathbf{Q} + s_0\mathbf{E}.$$

Also, $\mathbf{D} = \mathbf{B} - \mathbf{A}$ for some position vectors of M,

$$\mathbf{A} = \mathbf{Q} + s_1\mathbf{E}, \qquad \mathbf{B} = \mathbf{Q} + s_2\mathbf{E}.$$

Thus $$\mathbf{D} = (\mathbf{Q} + s_2\mathbf{E}) - (\mathbf{Q} + s_1\mathbf{E}) = (s_2 - s_1)\mathbf{E}.$$

Since $\mathbf{D} \neq 0$, $s_2 - s_1 \neq 0$. Thus the following are equivalent:

$$\begin{aligned}
&\mathbf{X} = \mathbf{P} + t\mathbf{D} && \text{for some } t,\\
&\mathbf{X} = \mathbf{Q} + s_0\mathbf{E} + t(s_2 - s_1)\mathbf{E} && \text{for some } t,\\
&\mathbf{X} = \mathbf{Q} + (s_0 + ts_2 - ts_1)\mathbf{E} && \text{for some } t,\\
&\mathbf{X} = \mathbf{Q} + s\mathbf{E} && \text{for some } s.
\end{aligned}$$

COROLLARY 2

Two points in space determine a line. The line through A and B has the vector equation

$$\mathbf{X} = \mathbf{A} + t(\mathbf{B} - \mathbf{A}).$$

PROOF A line L passes through A and B if and only if $\mathbf{A}$ is a position vector of L and $\mathbf{B} - \mathbf{A}$ is a direction vector of L. By Theorem 2, this happens if and only if L is the line with the vector equation $\mathbf{X} = \mathbf{A} + t(\mathbf{B} - \mathbf{A})$.

EXAMPLE 5 Find a vector equation of the line through the points

$$A(3, -4, 2), \qquad B(0, 8, 1).$$

The line has the equation

$$\mathbf{X} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} + t((0 - 3)\mathbf{i} + (8 - (-4))\mathbf{j} + (1 - 2)\mathbf{k}),$$
$$\mathbf{X} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} + t(-3\mathbf{i} + 12\mathbf{j} - \mathbf{k}).$$

The formula $\frac{1}{2}(\mathbf{A} + \mathbf{B})$ for the midpoint of the line segment AB holds for three as well as two dimensions.

EXAMPLE 6 Find the midpoint of the line segment AB where

$$A = (1, 4, -6), \qquad B = (2, 6, 0).$$

The midpoint C has position vector

$$\mathbf{C} = \tfrac{1}{2}[(\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}) + (2\mathbf{i} + 6\mathbf{j})] = \tfrac{3}{2}\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}.$$

Thus $C = (\frac{3}{2}, 5, -3)$.

PROBLEMS FOR SECTION 10.3

In Problems 1–3, find the vector represented by the directed line segment $\overrightarrow{PQ}$.

1 $P = (0, 0, 1), \quad Q = (5, -1, 8)$

2 $P = (5, 10, 0), \quad Q = (4, 10, 1)$

3 $P = (7, -2, 4), \quad Q = (7, -2, 3)$

In Problems 4–6, find the point Q such that $\mathbf{A}$ is the vector from P to Q.

4 $P = (4, 6, -4), \quad \mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

5 $P = (1, -2, 3), \quad \mathbf{A} = -\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

6 $P = (0, 0, 0), \quad \mathbf{A} = -3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

In Problems 7–22, find the given vector or scalar where

$$\mathbf{A} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}, \qquad \mathbf{B} = 2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$$

7 $\mathbf{A} + \mathbf{B}$

8 $\mathbf{A} - \mathbf{B}$

9 $\mathbf{B} - \mathbf{A}$

10 $4\mathbf{A}$

11 $-\mathbf{B}$

12 $-3\mathbf{A} + 4\mathbf{B}$

13 $|\mathbf{A}|$

14 $|\mathbf{B}|$

15 $|\mathbf{A} + \mathbf{B}|$

16 $|\mathbf{B} - \mathbf{A}|$

17 The angle between $\mathbf{A}$ and $\mathbf{B}$

18 The angle between $\mathbf{A}$ and $\mathbf{A} + \mathbf{B}$

19 The angle between $\mathbf{A}$ and $3\mathbf{A}$

20 The angle between $\mathbf{A}$ and $-2\mathbf{A}$

21 The unit vector and direction cosines of $\mathbf{A}$

22 The unit vector and direction cosines of $\mathbf{B}$

23 Find the vector with length 6 and direction cosines $(-1/2, 1/2, 1/\sqrt{2})$.

24 Find the vector with length $\sqrt{3}$ and direction cosines $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

25 If $\frac{1}{3}$ and $\frac{2}{3}$ are two of the direction cosines of a vector, what are the two possible values for the third direction cosine?

26 If the three forces

$$\mathbf{F}_1 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \qquad \mathbf{F}_2 = 3\mathbf{i} - \mathbf{j} - \mathbf{k}, \qquad \mathbf{F}_3 = 4\mathbf{k}$$

are acting on an object, find the total force.

27 If a force $\mathbf{F} = 6\mathbf{i} - 10\mathbf{j} + 2\mathbf{k}$ is acting on an object of mass 20, find its acceleration vector.

28 If a trader has the initial commodity vector $\mathbf{A} = 15\mathbf{i} + 20\mathbf{j} + 30\mathbf{k}$ and buys the commodity vector $\mathbf{B} = 2\mathbf{i} + \mathbf{k}$, find his new commodity vector.

29 If three commodities have the original price vector $\mathbf{P} = 100\mathbf{i} + 200\mathbf{j} + 500\mathbf{k}$ and all prices increase 25%, find the new price vector.

In Problems 30–35, find a vector equation for the given line.

30 The line with parametric equations $x = -t$, $y = 1 + \sqrt{2}t$, $z = 6 - 8t$.

31 The line with parametric equations $x = 1 + t$, $y = 3$, $z = 1 - t$.

32 The line through the points $P(0, 0, 0)$, $Q(1, 2, 3)$.

33 The line through the points $P(-1, 4, 3)$, $Q(-2, -3, 6)$.

34 The line through the point $P(4, 4, 5)$ with direction cosines $(1/\sqrt{6}, \sqrt{2}/\sqrt{6}, \sqrt{3}/\sqrt{6})$.

35 The line through the origin with direction cosines $(-\frac{3}{5}, 0, \frac{4}{5})$.

36 Find the midpoint of the line segment AB where $A = (-6, 3, 1)$, $B = (0, -4, 0)$.

37 Find the midpoint of AB where $A = (1, 2, 3)$, $B = (-1, 2, 7)$.

38 Find the midpoint of AB where $A = (6, 8, 10)$, $B = (-6, -8, -10)$.

39 Prove that if two sides of a triangle in space have equal lengths, then the angles opposite them are equal.

□ 40 Prove that if θ is the angle between $\mathbf{A}$ and $\mathbf{B}$ then $\pi - \theta$ is the angle between $\mathbf{A}$ and $-\mathbf{B}$. *Hint:* Show that the sum of the cosines is zero.

□ 41 Prove the Triangle Inequality for three dimensions.

□ 42 Use the Triangle Inequality to prove that if $\mathbf{A}$ and $\mathbf{B}$ are two nonzero vectors then

$$-1 \le \frac{|\mathbf{A}|^2 + |\mathbf{B}|^2 - |\mathbf{B} - \mathbf{A}|^2}{2|\mathbf{A}||\mathbf{B}|} \le 1.$$

10.4 PRODUCTS OF VECTORS

In the preceding sections we studied the sum of two vectors and the product of a scalar and a vector. We shall now define the inner product (or scalar or dot product) of two vectors $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \cdot \mathbf{B}$.

The inner product arises in quite different ways in physics and economics. We first discuss an example from economics.

If the price per unit of a commodity is p, the cost of a units of the commodity is the product pa. Similarly, if a pair of commodities has price vector

$$\mathbf{P} = p_1\mathbf{i} + p_2\mathbf{j},$$

the cost of a commodity vector

$$\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j}$$

is found by adding the products of the prices and quantities,

$$\text{cost} = p_1a_1 + p_2a_2.$$

If three commodities have price vector

$$\mathbf{P} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k},$$

the cost of a commodity vector

$$\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

is the sum of products,

$$\text{cost} = p_1a_1 + p_2a_2 + p_3a_3.$$

Notice that the cost is always a scalar. The quantity

$$p_1a_1 + p_2a_2 + p_3a_3$$

is the inner product of the vectors **P** and **A**.

DEFINITION

Two Dimensions *The **inner product** of* $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j}$ *and* $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j}$ *is the scalar*

$$\mathbf{A} \cdot \mathbf{B} = a_1b_1 + a_2b_2.$$

Three Dimensions *The **inner product** of* $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ *and* $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ *is the scalar*

$$\mathbf{A} \cdot \mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3.$$

Thus the cost of a commodity vector **A** at the price vector **P** is equal to the inner product of **P** and **A**, cost $= \mathbf{P} \cdot \mathbf{A}$.

EXAMPLE 1 Compute the inner product of $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{j} + \mathbf{k}$.

$$(\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \cdot (\mathbf{j} + \mathbf{k}) = 1 \cdot 0 + (-1) \cdot 1 + 3 \cdot 1 = 2.$$

EXAMPLE 2 Find the cost of one unit of commodity a, 3 units of commodity b, and 2 units of commodity c if the prices per unit are 6, 4, and 10 respectively.

$$\begin{aligned} \text{cost} &= (6\mathbf{i} + 4\mathbf{j} + 10\mathbf{k}) \cdot (\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) \\ &= 6 \cdot 1 + 4 \cdot 3 + 10 \cdot 2 = 38. \end{aligned}$$

EXAMPLE 3 Suppose a trader buys a commodity vector

$$\mathbf{A} = 40\mathbf{i} + 60\mathbf{j} + 100\mathbf{k}$$

at the price vector

$$\mathbf{P} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

and then sells it at the new price vector

$$\mathbf{Q} = 2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}.$$

Find his profit (or loss).

Since the trader pays $\mathbf{P} \cdot \mathbf{A}$ and receives $\mathbf{Q} \cdot \mathbf{A}$, his profit is given by

$$\text{profit} = \mathbf{Q} \cdot \mathbf{A} - \mathbf{P} \cdot \mathbf{A}.$$

Thus

$$\begin{aligned} \text{profit} &= (2 \cdot 40 + 5 \cdot 60 + 3 \cdot 100) - (3 \cdot 40 + 2 \cdot 60 + 4 \cdot 100) \\ &= 40. \end{aligned}$$

A positive number indicates a profit and a negative number indicates a loss.

EXAMPLE 4 A buyer has \$7500 and plans to buy a commodity vector **B** in the direction of the unit vector

$$\mathbf{U} = \tfrac{2}{3}\mathbf{i} + \tfrac{2}{3}\mathbf{j} + \tfrac{1}{3}\mathbf{k}.$$

Find the largest such commodity vector **B** which he can buy if the price vector is

$$\mathbf{P} = 2\mathbf{i} + 5\mathbf{j} + \mathbf{k}.$$

We must have $\mathbf{B} = t\mathbf{U}$ for some positive t, and also

$$\mathbf{P} \cdot \mathbf{B} = 7500.$$

We solve for t.

$$7500 = \mathbf{P} \cdot \mathbf{B} = \mathbf{P} \cdot t\mathbf{U} = t(\mathbf{P} \cdot \mathbf{U}).$$

$$t = \frac{7500}{\mathbf{P} \cdot \mathbf{U}} = \frac{7500}{2 \cdot \frac{2}{3} + 5 \cdot \frac{2}{3} + 1 \cdot \frac{1}{3}} = \frac{7500}{5} = 1500.$$

Thus $$\mathbf{B} = t\mathbf{U} = 1000\mathbf{i} + 1000\mathbf{j} + 500\mathbf{k}.$$

Another illustration of an inner product is the notion of *work* in physics. Suppose a force vector **F** acts on an object which moves in a straight line with a displacement vector **S**. If the force **F** has the same direction as the displacement **S**, i.e., the angle θ between **F** and **S** is zero, work is simply the product of the magnitudes of **F** and **S**;

$$W = |\mathbf{F}||\mathbf{S}| \qquad \text{if } \theta = 0.$$

In general, work depends on the *component* of the force in the direction of the displacement, that is, the product $|\mathbf{F}| \cos \theta$. The geometric meaning of this component is shown in Figure 10.4.1.

Work is defined as the product of the component of force in the direction of **S** and the length of **S**, so

$$W = |\mathbf{F}||\mathbf{S}| \cos \theta.$$

Work is thus a scalar quantity. It is positive if the angle θ is less than 90°, zero if $\theta = 90°$, and negative if $\theta > 90°$. Our first theorem shows that work is equal to the inner product of **F** and **S**,

$$W = \mathbf{F} \cdot \mathbf{S}.$$

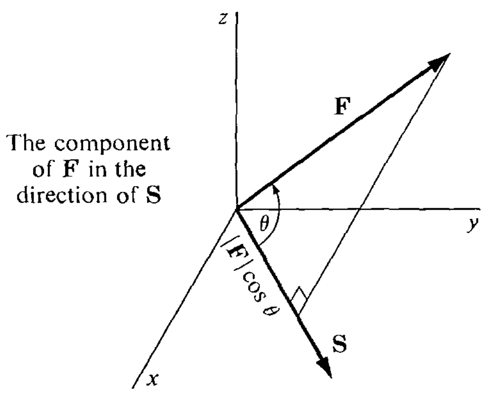


Figure 10.4.1

THEOREM 1

If θ is the angle between two nonzero vectors **A** *and* **B**, *then* $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos \theta$.

PROOF We give the proof in two dimensions. By the Law of Cosines,

$$\cos \theta = \frac{|\mathbf{A}|^2 + |\mathbf{B}|^2 - |\mathbf{B} - \mathbf{A}|^2}{2|\mathbf{A}||\mathbf{B}|}$$

$$= \frac{(a_1^2 + a_2^2) + (b_1^2 + b_2^2) - [(b_1 - a_1)^2 + (b_2 - a_2)^2]}{2|\mathbf{A}||\mathbf{B}|}$$

$$= \frac{2b_1a_1 + 2b_2a_2}{2|\mathbf{A}||\mathbf{B}|} = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|}.$$

Multiplying through by $|\mathbf{A}||\mathbf{B}|$, we have

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos \theta.$$

EXAMPLE 5 A lawnmower is moved horizontally (in the x direction) a distance of 10 feet. Find the work done if the lawnmower is pushed by a force F where

(a) $|\mathbf{F}| = 15$ pounds, $\theta = 30°$. (See Figure 10.4.2a.)
(b) $\mathbf{F} = 8\mathbf{i} - 5\mathbf{j}$, in pounds. (See Figure 10.4.2b.)

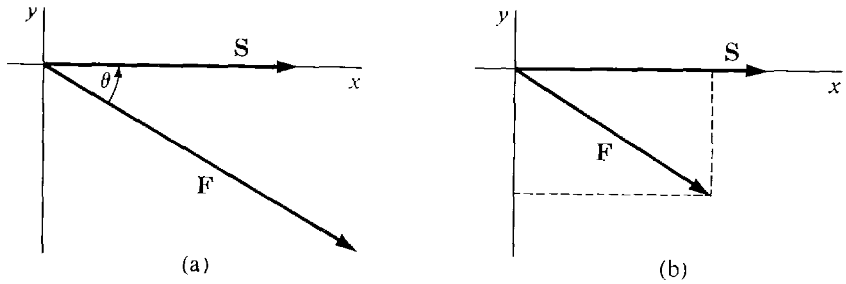


Figure 10.4.2

(a) $\cos\theta = \frac{1}{2}\sqrt{3}$. $|\mathbf{S}| = 10$.

$$W = |\mathbf{F}||\mathbf{S}| \cos \theta = 15 \cdot 10 \cdot \tfrac{1}{2}\sqrt{3} = 75\sqrt{3} \text{ ft lbs.}$$

(b) $W = \mathbf{F} \cdot \mathbf{S} = 8 \cdot 10 + (-5) \cdot 0 = 80$ ft lbs.

The angle between two vectors can be easily computed using the inner product.

COROLLARY

If **A** *and* **B** *are nonzero vectors, the angle θ between them has cosine*

$$\cos \theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|}.$$

EXAMPLE 6 Find the angle between the vectors

$$\mathbf{A} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}, \qquad \mathbf{B} = -\mathbf{i} + 5\mathbf{j} + \mathbf{k}.$$

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{3(-1) + 1\cdot 5 + (-1)\cdot 1}{\sqrt{3^2+1^2+1^2}\sqrt{1^2+5^2+1^2}} = \frac{1}{\sqrt{11\cdot 27}}.$$

$$\theta = \arccos\frac{1}{\sqrt{11\cdot 27}}.$$

Here is a list of algebraic rules for inner products. All the rules are easy to prove in either two or three dimensions.

THEOREM 2 (Algebraic Rules for Inner Products)

(i) $\mathbf{A}\cdot\mathbf{i} = a_1, \quad \mathbf{A}\cdot\mathbf{j} = a_2, \quad \mathbf{A}\cdot\mathbf{k} = a_3.$

(ii) $\mathbf{A}\cdot\mathbf{0} = \mathbf{0}\cdot\mathbf{A} = 0.$

(iii) $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$ (*Commutative Law*).

(iv) $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$ (*Distributive Law*).

(v) $(t\mathbf{A})\cdot\mathbf{B} = t(\mathbf{A}\cdot\mathbf{B})$ (*Associative Law*).

(vi) $\mathbf{A}\cdot\mathbf{A} = |\mathbf{A}|^2.$

PROOF Rule (vi) is proved as follows in three dimensions.

$$\mathbf{A}\cdot\mathbf{A} = a_1a_1 + a_2a_2 + a_3a_3 = a_1^2 + a_2^2 + a_3^2 = |\mathbf{A}|^2.$$

Inner products are useful in the study of perpendicular and parallel vectors.

DEFINITION

Two nonzero vectors **A** *and* **B** *are said to be* ***perpendicular*** *(or* ***orthogonal****),* $\mathbf{A}\perp\mathbf{B}$*, if the angle between them is* $\pi/2$*.* **A** *and* **B** *are said to be* ***parallel****,* $\mathbf{A}\parallel\mathbf{B}$*, if the angle between them is either* 0 *or* π*.*

TEST FOR PERPENDICULARS

Let **A** *and* **B** *be nonzero vectors. Then* $\mathbf{A}\perp\mathbf{B}$ *if and only if* $\mathbf{A}\cdot\mathbf{B} = 0$.

PROOF The following are equivalent:

$$\mathbf{A}\cdot\mathbf{B} = 0, \qquad \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = 0, \qquad \cos\theta = 0, \qquad \theta = \pi/2.$$

TEST FOR PARALLELS

Given two nonzero vectors **A** *and* **B**, *the following are equivalent:*

(i) $\mathbf{A}\parallel\mathbf{B}$.

(ii) $|\mathbf{A}\cdot\mathbf{B}| = |\mathbf{A}||\mathbf{B}|$.

(iii) **A** *is a scalar multiple of* **B**.

PROOF To show that (i) is equivalent to (ii), we note that the following are equivalent.

$$\mathbf{A} \parallel \mathbf{B},$$
$$\cos\theta = \pm 1.$$
$$\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \pm 1.$$
$$\mathbf{A}\cdot\mathbf{B} = \pm|\mathbf{A}||\mathbf{B}|.$$
$$|\mathbf{A}\cdot\mathbf{B}| = |\mathbf{A}||\mathbf{B}|.$$

We now show that (i) implies (iii), and (iii) implies (i). Assume (i), $\mathbf{A}\|\mathbf{B}$.

Case 1 $\theta = 0$. Let $\mathbf{U}$ and $\mathbf{V}$ be the unit vectors of $\mathbf{A}$ and $\mathbf{B}$. By the Law of Cosines,

$$\cos\theta = \frac{|\mathbf{U}|^2 + |\mathbf{V}|^2 - |\mathbf{V}-\mathbf{U}|^2}{2|\mathbf{U}||\mathbf{V}|},$$
$$1 = \frac{2 - |\mathbf{V}-\mathbf{U}|^2}{2},$$
$$|\mathbf{V}-\mathbf{U}| = 0,$$
$$\frac{\mathbf{A}}{|\mathbf{A}|} = \mathbf{U} = \mathbf{V} = \frac{\mathbf{B}}{|\mathbf{B}|},$$
$$\mathbf{A} = \frac{|\mathbf{A}|}{|\mathbf{B}|}\mathbf{B}.$$

Case 2 $\theta = \pi$. We see, by a similar proof, that $\mathbf{A} = -\frac{|\mathbf{A}|}{|\mathbf{B}|}\mathbf{B}$. In either case, $\mathbf{A}$ is a scalar multiple of $\mathbf{B}$.

Finally, assume (iii), say $\mathbf{A} = t\mathbf{B}$. Then

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{t\mathbf{B}\cdot\mathbf{B}}{|t\mathbf{B}||\mathbf{B}|} = \frac{t(\mathbf{B}\cdot\mathbf{B})}{|t||\mathbf{B}||\mathbf{B}|} = \frac{t|\mathbf{B}|^2}{|t||\mathbf{B}|^2} = \pm 1.$$

Therefore $\theta = 0$ or $\theta = \pi$, so $\mathbf{A}\|\mathbf{B}$.

EXAMPLE 7 Test for $\mathbf{A} \perp \mathbf{B}$ and $\mathbf{A} \parallel \mathbf{B}$ using the inner product.

(a) $\mathbf{A} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}, \quad \mathbf{B} = \mathbf{i} - 3\mathbf{j} + \mathbf{k}.$

We compute $\mathbf{A}\cdot\mathbf{B}$ and $|\mathbf{A}||\mathbf{B}|$.

$$\mathbf{A}\cdot\mathbf{B} = -1, \qquad |\mathbf{A}||\mathbf{B}| = 11.$$

Since $\mathbf{A}\cdot\mathbf{B} \neq 0$, not $\mathbf{A} \perp \mathbf{B}$.

Since $\mathbf{A}\cdot\mathbf{B} \neq \pm|\mathbf{A}||\mathbf{B}|$, not $\mathbf{A}\|\mathbf{B}$

(b) $\mathbf{A} = 2\mathbf{i} - \sqrt{3}\mathbf{j} + \mathbf{k}, \quad \mathbf{B} = -\sqrt{8}\mathbf{i} + \sqrt{6}\mathbf{j} - \sqrt{2}\mathbf{k}.$

$$\mathbf{A}\cdot\mathbf{B} = -8\sqrt{2}, \qquad |\mathbf{A}||\mathbf{B}| = 8\sqrt{2}.$$

Therefore $\mathbf{A} \parallel \mathbf{B}$.

(c) $\mathbf{A} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}, \quad \mathbf{B} = \mathbf{i} - 3\mathbf{j}.$

$\mathbf{A}\cdot\mathbf{B} = 0$. Therefore $\mathbf{A} \perp \mathbf{B}$.

Figure 10.4.3 illustrates this example.

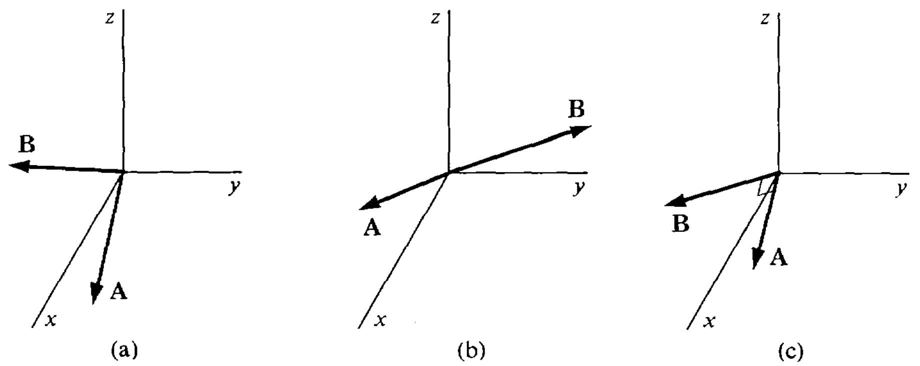


Figure 10.4.3

We conclude this section with a theorem about perpendicular vectors, first in the plane and then in space.

THEOREM 3

Let $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j}$ *be a nonzero vector in the plane.*

(i) *The vector* $\mathbf{B} = a_2\mathbf{i} - a_1\mathbf{j}$ *is perpendicular to* $\mathbf{A}$.

(ii) *Any vector perpendicular to* $\mathbf{A}$ *is parallel to* $\mathbf{B}$.

PROOF (i) We compute $\mathbf{A} \cdot \mathbf{B} = a_1a_2 + a_2(-a_1) = 0$.

(ii) If $\mathbf{C} \perp \mathbf{A}$, then both $\mathbf{B}$ and $\mathbf{C}$ make angles of $\pi/2$ with $\mathbf{A}$, so the angle between $\mathbf{B}$ and $\mathbf{C}$ is either 0 or π. Therefore $\mathbf{B} \parallel \mathbf{C}$.

EXAMPLE 8 Find a vector perpendicular to $\mathbf{A} = 4\mathbf{i} - 7\mathbf{j}$.

Answer $\mathbf{B} = -7\mathbf{i} - 4\mathbf{j}$ (Figure 10.4.4).

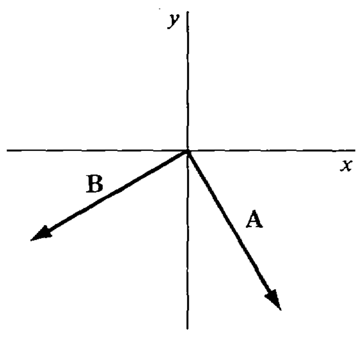


Figure 10.4.4

Theorem 3 raises the following problem about vectors in space. *Given two vectors* $\mathbf{A}$ *and* $\mathbf{B}$ *which are neither zero nor parallel, find a third vector* $\mathbf{C}$ *which is perpendicular to both* $\mathbf{A}$ *and* $\mathbf{B}$. For example, if $\mathbf{A}$ is $\mathbf{i}$ and $\mathbf{B}$ is $\mathbf{j}$, then the vector $\mathbf{k}$ is perpendicular to both $\mathbf{i}$ and $\mathbf{j}$. So is any scalar multiple of $\mathbf{k}$. In general it is not easy to see how to find a vector perpendicular to both $\mathbf{A}$ and $\mathbf{B}$. In fact, to solve the problem we need a new kind of product of vectors, the vector product

$$\mathbf{A} \times \mathbf{B}.$$

DEFINITION

Given two vectors

$$\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \qquad \mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

in space, the ***vector product*** *(or* ***cross product****) is the new vector*

$$\mathbf{A} \times \mathbf{B} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

This definition can be remembered by writing down the determinant

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

The positive and negative terms of $\mathbf{A} \times \mathbf{B}$ are the products of the diagonals shown in Figure 10.4.5.

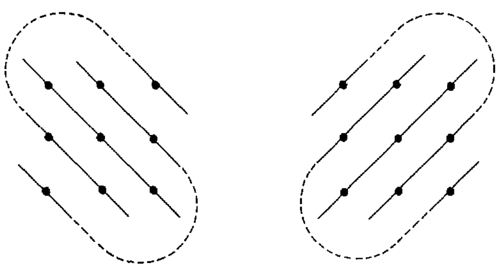


Figure 10.4.5 Positive terms Negative terms

EXAMPLE 9 Find $\mathbf{A} \times \mathbf{B}$ where

$$\mathbf{A} = 4\mathbf{i} - \mathbf{j} + \mathbf{k}, \qquad \mathbf{B} = 2\mathbf{j} - \mathbf{k}.$$

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 1 \\ 0 & 2 & -1 \end{vmatrix} \\ &= ((-1)(-1) - 1 \cdot 2)\mathbf{i} + (1 \cdot 0 - 4(-1))\mathbf{j} + (4 \cdot 2 - (-1) \cdot 0)\mathbf{k} \\ &= -\mathbf{i} + 4\mathbf{j} + 8\mathbf{k}.\end{aligned}$$

A, **B** and $\mathbf{A} \times \mathbf{B}$ are shown in Figure 10.4.6.

The vector products of the unit vectors **i**, **j**, and **k** are

$$\begin{array}{lll} \mathbf{i} \times \mathbf{i} = \mathbf{0}, & \mathbf{j} \times \mathbf{j} = \mathbf{0}, & \mathbf{k} \times \mathbf{k} = \mathbf{0}, \\ \mathbf{i} \times \mathbf{j} = \mathbf{k}, & \mathbf{j} \times \mathbf{k} = \mathbf{i}, & \mathbf{k} \times \mathbf{i} = \mathbf{j}, \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k}, & \mathbf{k} \times \mathbf{j} = -\mathbf{i}, & \mathbf{i} \times \mathbf{k} = -\mathbf{j}. \end{array}$$

Notice that $\mathbf{A} \cdot \mathbf{B}$ is a scalar but $\mathbf{A} \times \mathbf{B}$ is a vector.

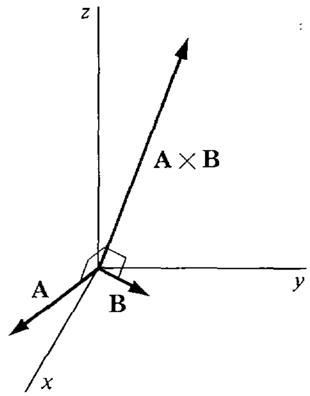


Figure 10.4.6

THEOREM 4

Let **A** *and* **B** *be two vectors in space which are not zero and not parallel.*

(i) **A** × **B** *is perpendicular to both* **A** *and* **B**.
(ii) *Any vector perpendicular to both* **A** *and* **B** *is parallel to* **A** × **B**.

PROOF (i) We compute the inner products.

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B}) &= a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1)\\ &= a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_1a_2b_3 + a_1a_3b_2 - a_2a_3b_1\\ &= 0.\end{aligned}$$

Similarly $\mathbf{B}\cdot(\mathbf{A}\times\mathbf{B}) = 0$.

It remains to prove that $\mathbf{A}\times\mathbf{B} \neq \mathbf{0}$. At least one component of **A**, say a_1, is nonzero. Let $t = b_1/a_1$ and let $\mathbf{C} = \mathbf{A}\times\mathbf{B}$. When we solve the equations

$$c_3 = a_1b_2 - a_2b_1, \qquad c_2 = a_3b_1 - a_1b_3$$

for b_2 and b_3, we get

$$b_1 = ta_1, \qquad b_2 = \frac{c_3}{a_1} + ta_2, \qquad b_3 = \frac{c_2}{a_1} + ta_3.$$

Since **B** is not parallel to **A**, $\mathbf{B} \neq t\mathbf{A}$. Therefore at least one of c_2, c_3 is nonzero, so $\mathbf{C} \neq \mathbf{0}$.

(ii) Let $\mathbf{C} = \mathbf{A}\times\mathbf{B}$ and let **D** be any other vector perpendicular to both **A** and **B**. Then

$$\mathbf{A}\cdot\mathbf{D} = a_1d_1 + a_2d_2 + a_3d_3 = 0.$$
$$\mathbf{B}\cdot\mathbf{D} = b_1d_1 + b_2d_2 + b_3d_3 = 0.$$

At least one component of **D**, say d_1, is nonzero. We may then solve the above equations for a_1 and b_1,

$$a_1 = -\frac{a_2d_2 + a_3d_3}{d_1}, \qquad b_1 = -\frac{b_2d_2 + b_3d_3}{d_1}.$$

Let $s = c_1/d_1$. Then $c_1 = sd_1$. Also,

$$c_2 = a_3b_1 - a_1b_3 = -\frac{a_3(b_2d_2 + b_3d_3)}{d_1} + \frac{b_3(a_2d_2 + a_3d_3)}{d_1}$$

$$= \frac{(a_2b_3 - a_3b_2)d_2}{d_1} = \frac{c_1d_2}{d_1} = sd_2.$$

Similarly, $c_3 = sd_3$. Therefore $\mathbf{C} = s\mathbf{D}$, and $\mathbf{C}$ is parallel to $\mathbf{D}$.

Warning: The Commutative Law and the Associative Law do *not* hold for the vector product. For example,

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \qquad \mathbf{j} \times \mathbf{i} = -\mathbf{k}$$

$$\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = \mathbf{0}, \qquad (\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = -\mathbf{i}.$$

However, vector products do satisfy the Distributive Laws

$$(s\mathbf{A} + t\mathbf{B}) \times \mathbf{C} = s(\mathbf{A} \times \mathbf{C}) + t(\mathbf{B} \times \mathbf{C}),$$

$$\mathbf{C} \times (s\mathbf{A} + t\mathbf{B}) = s(\mathbf{C} \times \mathbf{A}) + t(\mathbf{C} \times \mathbf{B}).$$

The proof is left as an exercise (Problem 36 at the end of this section).

Here is a brief summary of the operations on scalars and vectors.

Addition: $s + t$ is a scalar
$\mathbf{A} + \mathbf{B}$ is a vector
$s + \mathbf{A}$ is undefined

Multiplication: st is a scalar
$s\mathbf{A}$ is a vector
$\mathbf{A} \cdot \mathbf{B}$ is a scalar
$\mathbf{A} \times \mathbf{B}$ is a vector

Division: s/t is a scalar
$\mathbf{A}/t$ is a vector
$s/\mathbf{B}$ and $\mathbf{A}/\mathbf{B}$ are undefined

Absolute value and length: $|s|$ is a scalar
$|\mathbf{A}|$ is a scalar

One must be careful in forming longer expressions. For example,

$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C})$ is a scalar,
$(\mathbf{A} \cdot \mathbf{B}) + \mathbf{C}$ is undefined,
$(\mathbf{A} \cdot \mathbf{B})\mathbf{C}$ is a vector.

PROBLEMS FOR SECTION 10.4

In Problems 1–11, (a) compute $\mathbf{A} \cdot \mathbf{B}$, (b) test whether $\mathbf{A}$ is perpendicular or parallel to $\mathbf{B}$, and (c) find the cosine of the angle between $\mathbf{A}$ and $\mathbf{B}$ using $\mathbf{A} \cdot \mathbf{B}$.

1 $\mathbf{A} = \mathbf{i} - 3\mathbf{j}$, $\mathbf{B} = 2\mathbf{i} - 6\mathbf{j}$

2 $\mathbf{A} = 4\mathbf{i} - \mathbf{j}$, $\mathbf{B} = \mathbf{i} - \mathbf{j}$

3 $\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{B} = \mathbf{i} - \mathbf{k}$

4 $\mathbf{A} = \mathbf{i} + \mathbf{k}, \quad \mathbf{B} = \mathbf{j}$

5 $\mathbf{A} = 6\mathbf{i} - \mathbf{j}, \quad \mathbf{B} = 2\mathbf{i} - 12\mathbf{j}$

6 $\mathbf{A} = 5\mathbf{i} - \mathbf{j} + \mathbf{k}, \quad \mathbf{B} = -5\mathbf{i} + \mathbf{j} - \mathbf{k}$

7 $\mathbf{A} = \mathbf{i} + 4\mathbf{j} - 10\mathbf{k}, \quad \mathbf{B} = 4\mathbf{i} + \mathbf{j} + 10\mathbf{k}$

8 $\mathbf{A} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \quad \mathbf{B} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

9 $\mathbf{A} = \sqrt{2}\mathbf{i} + \sqrt{3}\mathbf{j} + \mathbf{k}, \quad \mathbf{B} = \sqrt{2}\mathbf{i} - \sqrt{3}\mathbf{j} + \mathbf{k}$

10 $\mathbf{A} = \sqrt{3}\mathbf{i} + \sqrt{6}\mathbf{j} - \mathbf{k}, \quad \mathbf{B} = \sqrt{3}\mathbf{i} - \sqrt{6}\mathbf{j} - 3\mathbf{k}$

11 $\mathbf{A} = \mathbf{i} - \sqrt{5}\mathbf{j} + \sqrt{2}\mathbf{k}, \quad \mathbf{B} = \sqrt{5}\mathbf{i} - 5\mathbf{j} + \sqrt{10}\mathbf{k}$

12 Which of the following are vectors, which are scalars, and which are undefined?

(a) $s(\mathbf{A} + \mathbf{B})$ (b) $(s + t)\mathbf{A}$

(c) $(s\mathbf{A}) \cdot (t\mathbf{B})$ (d) $s + (t\mathbf{A})$

(e) $s(\mathbf{A} \cdot \mathbf{B})$ (f) $\mathbf{A} + (\mathbf{B} \cdot \mathbf{C})$

(g) $\mathbf{A} \times (\mathbf{B} + \mathbf{C})$ (h) $\mathbf{A} \times (\mathbf{B} \cdot \mathbf{C})$

(i) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ (j) $(\mathbf{A} \times \mathbf{B}) + \mathbf{C}$

13 Find the cost of the commodity vector $\mathbf{A} = 15\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$ at the price vector $\mathbf{P} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

14 Find the profit or loss if a trader buys the commodity vector $\mathbf{A} = 3\mathbf{i} + 16\mathbf{j} + 4\mathbf{k}$ at the price vector $\mathbf{P} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$ and sells it at the price vector $3\mathbf{i} + 2\mathbf{j} + 10\mathbf{k}$.

15 A trader initially has the commodity vector $\mathbf{A} = \mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$. He sells his whole commodity vector at the price $\mathbf{P} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and uses the revenue from this sale to buy an equal amount of each commodity. Find his new commodity vector.

16 Find the amount of work done by the force vector $\mathbf{F} = 3\mathbf{i} - \mathbf{j} - 4\mathbf{k}$ acting along the displacement vector $\mathbf{S} = 5\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

17 Find the work done by a force vector of magnitude 10 acting along a displacement of length 40 if the angle between the force and displacement is 45°.

18 Prove that the basis vectors **i**, **j**, **k** are perpendicular.

19 Find a vector in the plane perpendicular to $\mathbf{A} = \mathbf{i} + \mathbf{j}$.

20 Find a vector in the plane perpendicular to $\mathbf{A} = 2\mathbf{i} - 9\mathbf{j}$.

21 Compute $\mathbf{A} \times \mathbf{B}$ where $\mathbf{A} = \mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{B} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$.

22 Compute $\mathbf{A} \times \mathbf{B}$ where $\mathbf{A} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{B} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

23 Compute $\mathbf{A} \times \mathbf{B}$ where $\mathbf{A} = \mathbf{i} + \mathbf{k}$, $\mathbf{B} = \mathbf{j} - \mathbf{k}$.

24 Find a vector perpendicular to both $\mathbf{A} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{B} = \mathbf{i} - \mathbf{j} + \mathbf{k}$.

25 Find a vector perpendicular to both $\mathbf{A} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{B} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$.

26 Find a vector perpendicular to both $\mathbf{A} = -\mathbf{i} - 4\mathbf{j} + \mathbf{k}$, $\mathbf{B} = \mathbf{j} - 2\mathbf{k}$.

27 Find a unit vector in the plane perpendicular to $\mathbf{A} = 3\mathbf{i} - 4\mathbf{j}$.

28 Find a unit vector in the plane perpendicular to $\mathbf{A} = 2\mathbf{i} - \mathbf{j}$.

29 Find a unit vector perpendicular to both $\mathbf{A} = \mathbf{i} + \mathbf{j}$, $\mathbf{B} = \mathbf{k}$.

30 Find a unit vector perpendicular to both $\mathbf{A} = 2\mathbf{i} + 3\mathbf{k}$, $\mathbf{B} = -\mathbf{i} + \mathbf{j} - \mathbf{k}$.

31 Find the angle between two long diagonals of a cube.

32 Find the angle between a long diagonal and a diagonal along a face of a cube.

33 Find the angle between the diagonals of two adjacent faces of a cube.

34 Show that the inner product of two unit vectors is equal to the cosine of the angle between them.

35 Use inner products to prove that the diagonals of a rhombus (a parallelogram whose sides have equal lengths) are perpendicular.

☐ 36 Prove the Distributive Law for vector products.

$$(s\mathbf{A} + t\mathbf{B}) \times \mathbf{C} = s(\mathbf{A} \times \mathbf{C}) + t(\mathbf{B} \times \mathbf{C}).$$

☐ 37 Prove the Anticommutative Law for vector products.

$$\mathbf{B} \times \mathbf{A} = -(\mathbf{A} \times \mathbf{B}).$$

☐ 38 Prove that $\mathbf{A} \parallel \mathbf{B}$ if and only if $\mathbf{A} \times \mathbf{B} = \mathbf{0}$ (where $\mathbf{A}, \mathbf{B}$ are nonzero).

☐ 39 Show that the length of $\mathbf{A} \times \mathbf{B}$ is equal to the area of the parallelogram with sides $\mathbf{A}$ and $\mathbf{B}$, in symbols

$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}| \sin \theta.$$

☐ 40 Prove that the "scalar triple product" $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is equal to the volume of a parallelopiped with edges $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$.

10.5 PLANES IN SPACE

DEFINITION

*A **plane** in space is the graph of an equation of the form*

$$ax + by + cz = d$$

where a, b, c are not all zero.

The simplest planes are those where two of the numbers a, b, c are zero.

The plane $x = d$ is parallel to the yz-plane.
The plane $y = d$ is parallel to the xz-plane.
The plane $z = d$ is parallel to the xy-plane.

These three cases are illustrated in Figure 10.5.1.

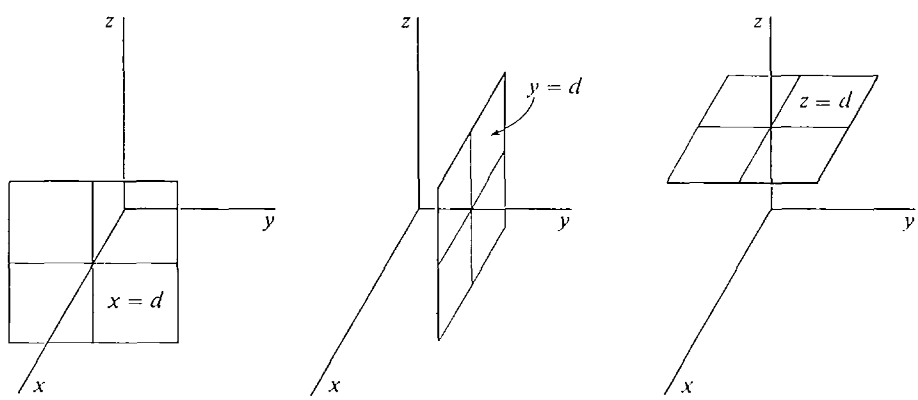


Figure 10.5.1

The examples below show how to draw sketches to help visualize other planes. The idea is to use the points where the plane cuts the coordinate axes, and to draw a triangular or rectangular portion of the plane.

EXAMPLE 1 (For sketching a plane where a, b, c and d are nonzero.) Sketch the plane $x + 2y + z = 2$.

Step 1 Find the points where the plane crosses the coordinate axes.

$$x\text{-axis:} \quad \text{When } y = z = 0, \quad x = 2.$$

The plane crosses the x-axis at $(2, 0, 0)$.

$$y\text{-axis:} \quad \text{When } x = z = 0, \quad y = 1.$$

The plane crosses the y-axis at $(0, 1, 0)$.

$$z\text{-axis:} \quad \text{When } x = y = 0, \quad z = 2.$$

The plane crosses the z-axis at $(0, 0, 2)$.

Step 2 Draw the triangle connecting these three points, as shown in Figure 10.5.2. This triangle lies in the plane.

EXAMPLE 2 (For sketching a plane where two of a, b, c are nonzero and $d \neq 0$.) Sketch the plane $2x + z = 4$.

Step 1 Find the points where the plane crosses the x- and z-axes.
The plane crosses the x-axis at $(2, 0, 0)$.
The plane crosses the z-axis at $(0, 0, 4)$.

Step 2 The plane is parallel to the y-axis. Draw a rectangle with two sides parallel to the y-axis and two sides parallel to the line segment from $(2, 0, 0)$ to $(0, 0, 4)$, as in Figure 10.5.3. This rectangle lies in the plane.

EXAMPLE 3 (For sketching a plane with $d = 0$.) Sketch the plane $x + 2y - z = 0$.

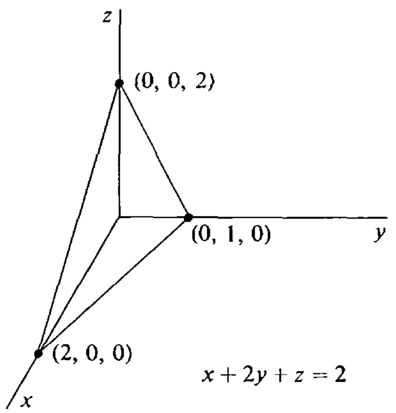


Figure 10.5.2

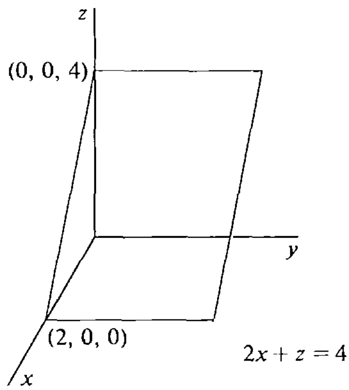


Figure 10.5.3

Step 1 The plane passes through the origin because $(0, 0, 0)$ is a solution of the equation. Find another point where $x = 0$ and a third point where y or $z = 0$,

$$x = 0, \quad y = 1, \quad z = 2,$$
$$x = 1, \quad y = 0, \quad z = 1.$$

Step 2 Connect the points $(0, 0, 0)$, $(0, 1, 2)$, $(1, 0, 1)$ to form a triangle which lies in the plane, as in Figure 10.5.4.

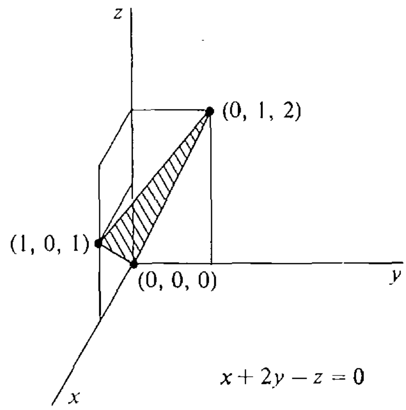


Figure 10.5.4

A *position vector* of a plane p is a vector **P** such that P is a point on the plane. A *direction vector* of p is a *vector* **D** from one point of p to another. A *normal vector* of p is a vector **N** which is perpendicular to every direction vector of p. These vectors are illustrated in Figure 10.5.5.

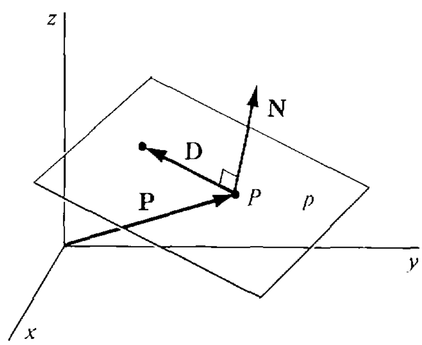


Figure 10.5.5 Position, Direction, and Normal Vectors

We shall often find it convenient to write a scalar equation

$$ax + by + cz = d$$

for a plane in vector form,

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \mathbf{X} = d.$$

We call this a *vector equation* for the plane.

THEOREM 1

(i) *A vector is normal to a plane* $\mathbf{N}\cdot\mathbf{X} = d$ *if and only if it is parallel to* $\mathbf{N}$.

(ii) *There is a unique plane with a given normal vector* $\mathbf{N}$ *and position vector* $\mathbf{P}$, *and it has the vector equation*

$$\mathbf{N}\cdot\mathbf{X} = \mathbf{N}\cdot\mathbf{P}.$$

PROOF (i) Call the plane p. For any direction vector $\mathbf{D} = \mathbf{Q} - \mathbf{P}$ of p, we have

$$\mathbf{N}\cdot\mathbf{D} = \mathbf{N}\cdot(\mathbf{Q} - \mathbf{P}) = \mathbf{N}\cdot\mathbf{Q} - \mathbf{N}\cdot\mathbf{P} = d - d = 0.$$

Let $\mathbf{M}$ be parallel to $\mathbf{N}$, say $\mathbf{M} = s\mathbf{N}$.

$$\mathbf{M}\cdot\mathbf{D} = (s\mathbf{N})\cdot\mathbf{D} = s(\mathbf{N}\cdot\mathbf{D}) = 0.$$

Hence $\mathbf{M} \perp \mathbf{D}$ and $\mathbf{M}$ is normal to p.

Now suppose $\mathbf{M}$ is normal to p. Let $\mathbf{C}$ and $\mathbf{D}$ be two nonparallel direction vectors of p. Then both $\mathbf{M}$ and $\mathbf{N}$ are perpendicular to $\mathbf{C}$ and $\mathbf{D}$. Therefore $\mathbf{M}$ and $\mathbf{N}$ are parallel to $\mathbf{C} \times \mathbf{D}$ and hence parallel to each other.

(ii) Set $d = \mathbf{N}\cdot\mathbf{P}$. The plane p with the equation $\mathbf{N}\cdot\mathbf{X} = d$ has position vector $\mathbf{P}$ and normal vector $\mathbf{N}$ by (i).

To show p is unique let q be any plane with position vector $\mathbf{P}$ and normal vector $\mathbf{N}$. q has a vector equation $\mathbf{M}\cdot\mathbf{X} = e$. By (i), $\mathbf{N}$ is parallel to $\mathbf{M}$, say $\mathbf{N} = s\mathbf{M}$. Then the following equations are equivalent for all $\mathbf{X}$:

$$\mathbf{N}\cdot\mathbf{X} = \mathbf{N}\cdot\mathbf{P} = d.$$
$$(s\mathbf{M})\cdot\mathbf{X} = (s\mathbf{M})\cdot\mathbf{P}.$$
$$s(\mathbf{M}\cdot\mathbf{X}) = s(\mathbf{M}\cdot\mathbf{P}).$$
$$\mathbf{M}\cdot\mathbf{X} = \mathbf{M}\cdot\mathbf{P} = e.$$

It follows that q equals p.

EXAMPLE 4 The plane $2x + 3y - z = 5$ has the normal vector

$$\mathbf{N} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$

and the vector equation

$$(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})\cdot\mathbf{X} = 5.$$

EXAMPLE 5 Find the vector and scalar equations for the plane with position and normal vectors

$$\mathbf{P} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}, \qquad \mathbf{N} = \mathbf{i} + \mathbf{j} + 4\mathbf{k}.$$

We first compute $\mathbf{N}\cdot\mathbf{P}$,

$$\mathbf{N}\cdot\mathbf{P} = 1\cdot 3 + 1\cdot(-1) + 4\cdot(-2) = -6.$$

A vector equation is $\quad (\mathbf{i} + \mathbf{j} + 4\mathbf{k})\cdot\mathbf{X} = -6.$

A scalar equation is $\quad x + y + 4z = -6.$

The plane is shown in Figure 10.5.6.

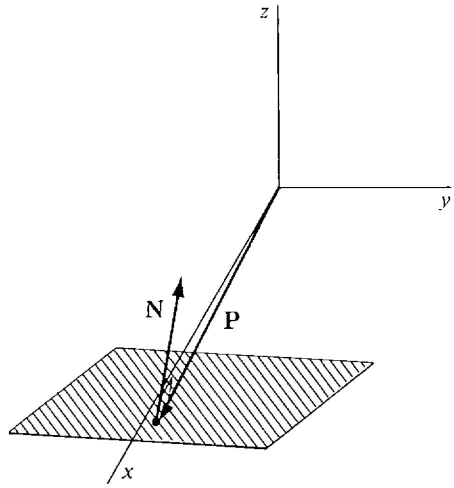


Figure 10.5.6

The vector product can sometimes be used to find a normal vector of a plane.

COROLLARY

If **C** *and* **D** *are two nonparallel direction vectors of a plane p, then* **C** × **D** *is a normal vector of p.*

PROOF p has some normal vector **N**. **N** is perpendicular to both **C** and **D**, and hence parallel to **C** × **D**, so **C** × **D** is a normal vector of p.

EXAMPLE 6 Find the plane with position vector $\mathbf{P} = \mathbf{k}$ and direction vectors $\mathbf{C} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\quad \mathbf{D} = -\mathbf{j}$.

First we find a normal vector of the plane,

$$\mathbf{N} = \mathbf{C} \times \mathbf{D} = (1 \cdot 0 - 1 \cdot (-1))\mathbf{i} + (1 \cdot 0 - (-2) \cdot 0)\mathbf{j} + ((-2)(-1) - 1 \cdot 0)\mathbf{k}$$
$$= \mathbf{i} + 2\mathbf{k}.$$

Then $$\mathbf{N} \cdot \mathbf{P} = 1 \cdot 0 + 0 \cdot 0 + 2 \cdot 1 = 2.$$

The plane has the vector equation $(\mathbf{i} + 2\mathbf{k}) \cdot \mathbf{X} = 2$

and the scalar equation $x + 2z = 2$.

The plane is shown in Figure 10.5.7.

EXAMPLE 7 Find the plane through the three points

$$P(-1, 3, 1), \qquad Q(1, 2, 3), \qquad S(-1, -1, 0).$$

The plane has position vector

$$\mathbf{P} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

and the two direction vectors

$$\mathbf{C} = \mathbf{Q} - \mathbf{P} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k},$$
$$\mathbf{D} = \mathbf{S} - \mathbf{P} = -4\mathbf{j} - \mathbf{k}.$$

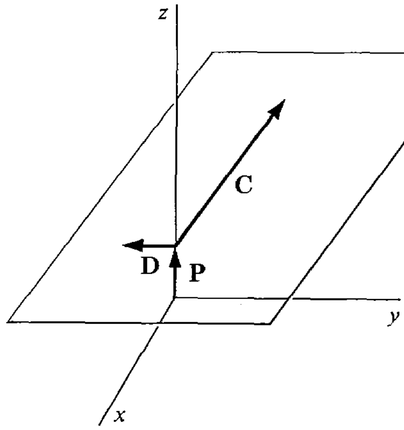


Figure 10.5.7

A normal vector of the plane is

$$\mathbf{N} = \mathbf{C} \times \mathbf{D} = ((-1)(-1) - 2(-4))\mathbf{i} + (2\cdot 0 - 2(-1))\mathbf{j} + (2(-4) - (-1)\cdot 0)\mathbf{k}$$
$$= 9\mathbf{i} + 2\mathbf{j} - 8\mathbf{k}.$$

Then $$\mathbf{N}\cdot\mathbf{P} = 9(-1) + 2\cdot 3 + (-8)\cdot 1 = -11.$$

The plane has the vector equation $(9\mathbf{i} + 2\mathbf{j} - 8\mathbf{k})\cdot\mathbf{X} = -11$

and the scalar equation $\quad 9x + 2y - 8z = -11.$

The plane is shown in Figure 10.5.8.

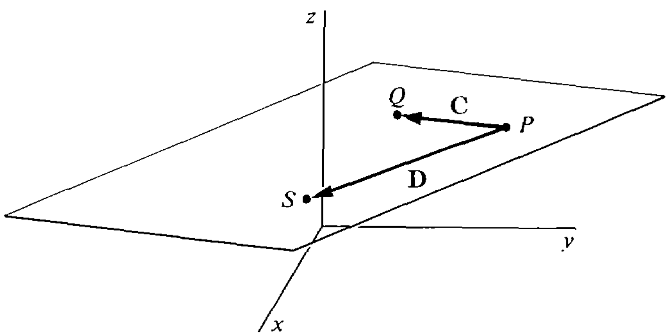


Figure 10.5.8

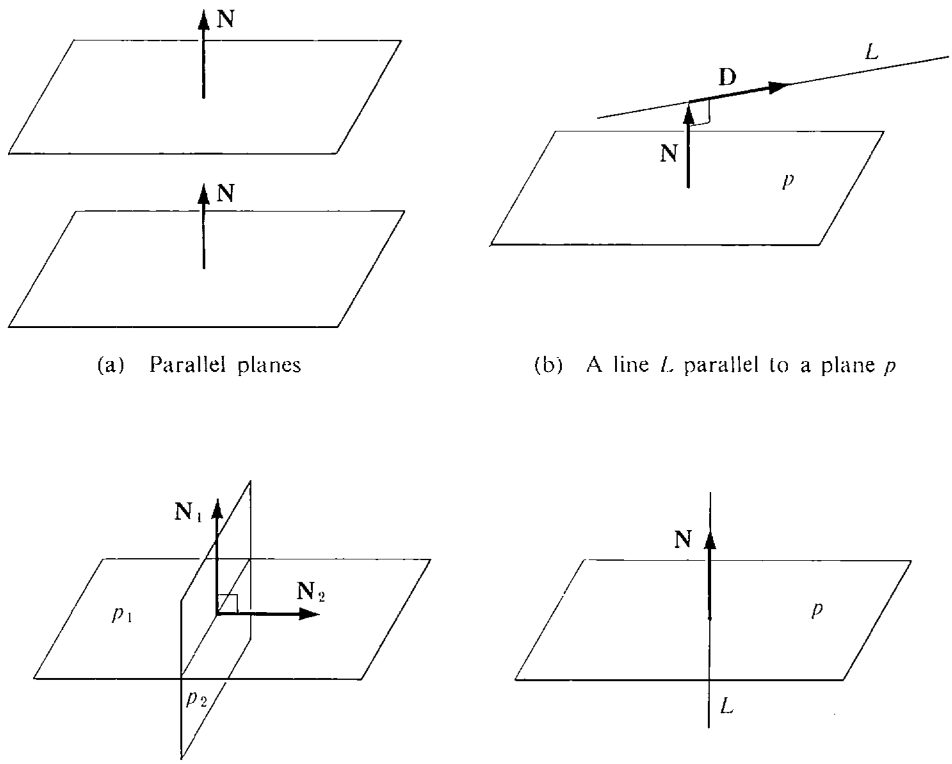
Two planes are said to be *parallel* if their normal vectors are parallel. A line L is said to be *parallel* to a plane p if the direction vectors of L are perpendicular to the normal vectors of p.

Two planes are said to be *perpendicular* if their normal vectors are perpendicular. A line L is said to be *perpendicular* to a plane p if the direction vectors of L are normal to p. Figure 10.5.9 illustrates these definitions.

EXAMPLE 8 Determine whether the plane $3x - 2y + z = 4$ and the line $\mathbf{X} = (3\mathbf{i} - \mathbf{j} + \mathbf{k}) + t(\mathbf{i} + \mathbf{j} - \mathbf{k})$ are parallel.

The plane has the normal vector $\quad \mathbf{N} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}.$

The line has the direction vector $\quad \mathbf{D} = \mathbf{i} + \mathbf{j} - \mathbf{k}.$



(a) Parallel planes (b) A line L parallel to a plane p

(c) Perpendicular planes (d) A line L perpendicular to a plane p

Figure 10.5.9

We compute $\quad \mathbf{N}\cdot\mathbf{D} = 3\cdot 1 + (-2)\cdot 1 + 1(-1) = 0.$

Therefore the plane and line are parallel (Figure 10.5.10).

EXAMPLE 9 Find the line L through the point $P(1, 2, 3)$ which is perpendicular to the plane $3x - 4y + z = 10$.

The plane has the normal vector $\mathbf{N} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$.

Therefore $\mathbf{N}$ is a direction vector of L, and L has the vector equation

$$\mathbf{X} = \mathbf{P} + t\mathbf{N},$$
$$= \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + t(3\mathbf{i} - 4\mathbf{j} + \mathbf{k})$$

(see Figure 10.5.11).

EXAMPLE 10 Find the plane p containing the line $\mathbf{X} = \mathbf{i} + t(\mathbf{j} + \mathbf{k})$ which is perpendicular to the plane $x + 3y - 2z = 0$.

The given plane q has the normal vector $\quad \mathbf{M} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k},$

and the given line L has the direction vector $\quad \mathbf{D} = \mathbf{j} + \mathbf{k}.$

The required plane p must have a normal vector $\mathbf{N}$ which is perpendicular to both $\mathbf{M}$ and $\mathbf{D}$, so we take

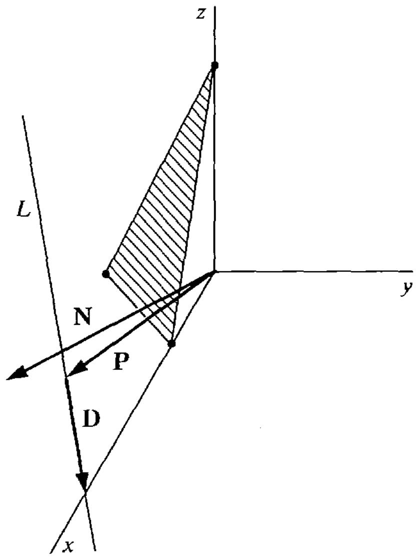


Figure 10.5.10

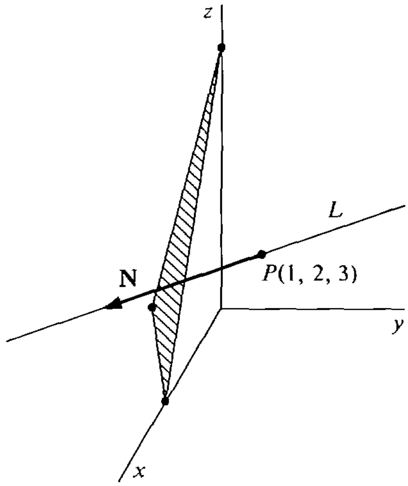


Figure 10.5.11

$$\mathbf{N} = \mathbf{M} \times \mathbf{D} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ 0 & 1 & 1 \end{vmatrix}, \qquad \mathbf{N} = 5\mathbf{i} - \mathbf{j} + \mathbf{k}.$$

The vector $\mathbf{P} = \mathbf{i}$ is a position vector of L and therefore a position vector of p. So p has the vector equation

$$\mathbf{N} \cdot \mathbf{X} = \mathbf{N} \cdot \mathbf{P},$$

$$(5\mathbf{i} - \mathbf{j} + \mathbf{k}) \cdot \mathbf{X} = 5,$$

and the scalar equation $\quad 5x - y + z = 5 \quad$ (see Figure 10.5.12).

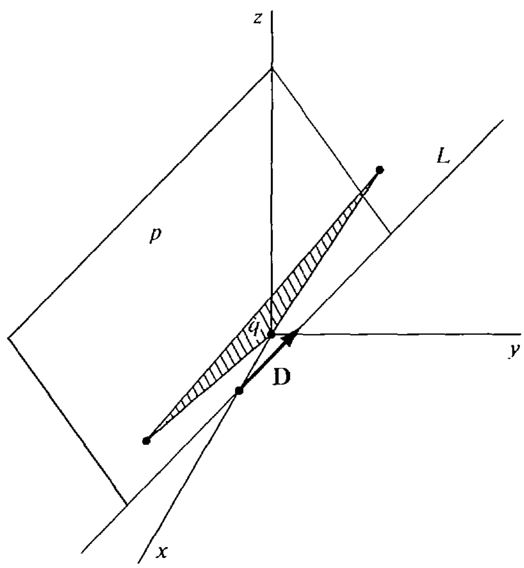


Figure 10.5.12

A line which is not parallel to a plane will intersect the plane at exactly one point.

EXAMPLE 11 Find the point at which the line $\mathbf{X} = \mathbf{i} - \mathbf{j} + \mathbf{k} + t(3\mathbf{i} - \mathbf{j} - \mathbf{k})$ intersects the plane $3x - 2y + z = 4$.

The line has the parametric equations

$$x = 1 + 3t, \qquad y = -1 - t, \qquad z = 1 - t.$$

We substitute these in the equation for the plane and solve for t.

$$3(1 + 3t) - 2(-1 - t) + (1 - t) = 4,$$
$$6 + 10t = 4,$$
$$t = -\tfrac{1}{5}.$$

Therefore the point of intersection is given by the parametric equations for the line at $t = -\frac{1}{5}$;

$$x = \tfrac{2}{5}, \qquad y = -\tfrac{4}{5}, \qquad z = \tfrac{6}{5},$$

(see Figure 10.5.13).

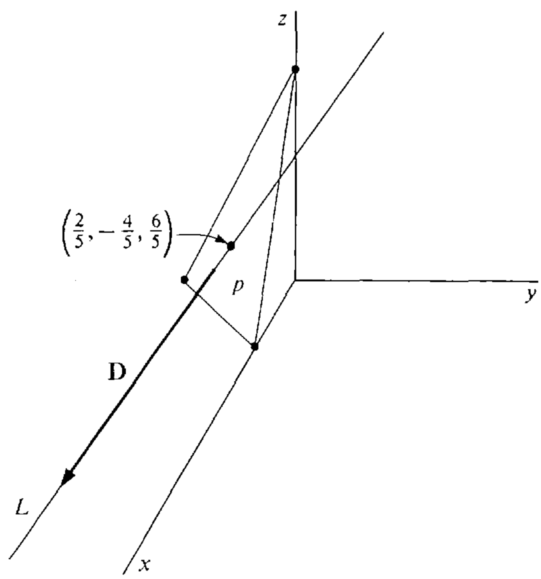


Figure 10.5.13

Two planes which are not parallel intersect at a line.

EXAMPLE 12 Find the line L of intersection of the planes

$$4x - 5y + z = 2,$$
$$x + 2z = 0.$$

Step 1 To get a position vector of L, we find any point on both planes. Setting $z = 0$ and solving for x and y, we obtain the point $S(0, -\frac{2}{5}, 0)$ on both planes. Thus $\mathbf{S} = -\frac{2}{5}\mathbf{j}$ is a position vector of L.

Step 2 To get a direction vector **D** of L we need a vector perpendicular to the normal vectors of both planes. The normal vectors are

$$\mathbf{M} = 4\mathbf{i} - 5\mathbf{j} + \mathbf{k}, \qquad \mathbf{N} = \mathbf{i} + 2\mathbf{k}.$$

We take
$$\mathbf{D} = \mathbf{M} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -5 & 1 \\ 1 & 0 & 2 \end{vmatrix}$$
$$= -10\mathbf{i} - 7\mathbf{j} + 5\mathbf{k}.$$

Thus L is the line $\quad \mathbf{X} = -\frac{2}{5}\mathbf{j} + t(-10\mathbf{i} - 7\mathbf{j} + 5\mathbf{k}).$

PROBLEMS FOR SECTION 10.5

In Problems 1–12 find the points, if any, where the plane meets the x, y, and z axes. and sketch the plane.

1	$x + y + z = 1$	2	$10x + 5y + z = 10$
3	$2x - 2y + z = 2$	4	$-x + 3y + 3z = -3$
5	$x + y = 1$	6	$y + 3z = 1$
7	$x - z = 2$	8	$x + y + z = 0$
9	$4x - y + 2z = 0$	10	$\frac{1}{2}x - y - z = 0$
11	$x - y = 0$	12	$2y + z = 0$

13 Find a normal vector to the following planes.

(a) $x - 3y + 6z = 4$ (b) $x + 2y = 0$
(c) $-3x + 4y + z = 0$ (d) $x + 6z = 8$
(e) $y = 4$ (f) $-y + z = 5$

Find a scalar equation for the plane described in Problems 14–32

14 The plane with normal vector $\mathbf{N} = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and position vector $\mathbf{P} = 2\mathbf{i} + \mathbf{k}$.

15 The plane with normal vector $\mathbf{N} = \mathbf{j} + 2\mathbf{k}$ and position vector $\mathbf{P} = \mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$.

16 The plane through the point (1, 5, 8) with normal vector $\mathbf{N} = 5\mathbf{i} + \mathbf{j} - \mathbf{k}$.

17 The plane through the origin with normal vector $\mathbf{N} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

18 The plane with position vector $\mathbf{P} = \mathbf{i} - \mathbf{j}$ and direction vectors $\mathbf{C} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{D} = \mathbf{i} - \mathbf{j} - \mathbf{k}$.

19 The plane through the point (1, 2, 3) with direction vectors $\mathbf{C} = \mathbf{i}, \mathbf{D} = \mathbf{j} + \mathbf{k}$.

20 The plane through the points $A(0, 4, 6)$, $B(5, 1, -1)$, $C(2, 6, 0)$.

21 The plane through the points $A(5, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, -4)$.

22 The plane through the points $A(4, 9, -6)$, $B(6, 6, 6)$, $C(1, 10, 0)$.

23 The plane through the point $A(1, 2, 4)$ containing the line
$$\mathbf{X} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} + t(\mathbf{i} - \mathbf{k}).$$

24 The plane through the point $A(0, 5, 1)$ containing the line
$$\mathbf{X} = \mathbf{i} + t(3\mathbf{i} - \mathbf{j} + \mathbf{k}).$$

25 The plane through the point $A(5, 0, 1)$ perpendicular to the line
$$\mathbf{X} = \mathbf{i} + \mathbf{j} + \mathbf{k} + t(2\mathbf{i} + \mathbf{j} + 3\mathbf{k}).$$

26 The plane through the origin perpendicular to the line
$$\mathbf{X} = t(5\mathbf{i} - \mathbf{j} + 6\mathbf{k}).$$

27 The plane through the point $A(4, 10, -3)$ parallel to the plane $x + y - 2z = 1$.

28 The plane through the origin parallel to the plane $4x + y + z = 6$.

29 The plane containing the line $\mathbf{X} = \mathbf{i} + \mathbf{j} - \mathbf{k} + t(3\mathbf{i} + \mathbf{k})$ and perpendicular to the plane $2x - y + z = 3$.

30 The plane containing the line $\mathbf{X} = 3\mathbf{j} + t(5\mathbf{i} + \mathbf{j} - 6\mathbf{k})$ and perpendicular to the plane $x + y + z = 0$.

31 The plane containing the line $\mathbf{X} = 3\mathbf{i} + \mathbf{j} + \mathbf{k} + t(\mathbf{i} - 6\mathbf{k})$ and parallel to the line $\mathbf{X} = \mathbf{i} + \mathbf{j} + t(3\mathbf{i} + 4\mathbf{j} + \mathbf{k})$.

32 The plane containing the x-axis and parallel to the line $\mathbf{X} = t(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$.

In Problems 33–36, test for perpendiculars and parallels.

33 The planes $x - 3y + 2z = 4$, $-2x + 6y - 4z = 0$.

34 The planes $4x + 3y - z = 6$, $x + y + 7z = 4$.

35 The plane $-x + y - 2z = 8$ and the line $\mathbf{X} = 2\mathbf{i} + \mathbf{k} + t(3\mathbf{i} - \mathbf{j} + \mathbf{k})$.

36 The plane $x + y + 3z = 10$ and the line $\mathbf{X} = 3\mathbf{j} + t(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$.

In Problems 37–42 find a vector equation for the given line.

37 The line through $P(5, 3, -1)$, perpendicular to the plane $x - y + 3z = 1$.

38 The line through the origin, perpendicular to the plane $x - y + z = 0$.

39 The line of intersection of the planes $x + y + z = 0$, $x - y + 2z = 1$.

40 The line of intersection of the planes $2x + 3y - 4z = 1$, $x + z = 4$.

41 The line of intersection of the planes $x + y = 1$, $y - z = 2$.

42 The line of intersection of the planes $x - 2y + 3z = 0$, $z = -2$.

In Problems 43–49, find the coordinates of the given point.

43 The point where the line $\mathbf{X} = 3\mathbf{i} + \mathbf{j} + \mathbf{k} + t(-\mathbf{i} + 3\mathbf{j} - \mathbf{k})$ intersects the plane $x + 2y - z = 4$.

44 The point where the line $\mathbf{X} = \mathbf{i} + \mathbf{k} + t(\mathbf{j} + \mathbf{k})$ intersects the plane $x + 2y = -3$.

45 The point where the line $\mathbf{X} = t(\mathbf{i} - 2\mathbf{k})$ intersects the plane $x - 3y + 2z = 4$.

46 The point P on the plane $x + 3y + 6z = 6$, nearest to the origin. *Hint:* The line from the origin to P must be perpendicular to the plane.

47 The point P on the plane $x + y + z = 1$, nearest to the point $A(-1, 2, 3)$.

48 The point P on the line $\mathbf{X} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + t(\mathbf{i} - \mathbf{j} + \mathbf{k})$ nearest to the origin. *Hint: P* must be on the plane through the origin perpendicular to the line.

49 The point P on the line $\mathbf{X} = \mathbf{j} + t(\mathbf{i} + 3\mathbf{k})$ nearest to the point $A(1, 2, 3)$.

□ 50 Prove that any three points which are not all on a line determine a plane.

□ 51 Prove that if a line and plane are parallel and have at least one point in common then the line is a subset of the plane.

□ 52 Prove that if two parallel planes have at least one point in common then they are equal.

□ 53 Let p be a plane with normal vector $\mathbf{N}$. Prove that every vector $\mathbf{D}$ perpendicular to $\mathbf{N}$ is a direction vector of p.

□ 54 Given a plane p and a line L not perpendicular to p, prove that there is a unique plane q which contains L and is perpendicular to p.

10.6 VECTOR VALUED FUNCTIONS

A vector valued function is a function **F** which maps real numbers to vectors. We shall study vector valued functions in either two or three dimensions. Here is the exact definition.

DEFINITION

A ***vector valued function*** *in two dimensions is a set* **F** *of ordered pairs* $(t, \mathbf{X})$ *such that for every real number t one of the following occurs.*

(i) *There is exactly one vector* **X** *in two dimensions for which the ordered pair* $(t, \mathbf{X})$ *belongs to* **F**. *In this case* $\mathbf{F}(t)$ *is defined and* $\mathbf{F}(t) = \mathbf{X}$.

(ii) *There is no* **X** *for which* $(t, \mathbf{X})$ *belongs to* **F**. *In this case* $\mathbf{F}(t)$ *is said to be undefined.*

The definition of a three-dimensional vector valued function is similar.

A vector valued function in two dimensions can be written as a sum

$$\mathbf{F}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j}.$$

The functions f_1 and f_2 are real functions of one variable, called the *components* of **F**. The *vector equation* $\mathbf{X} = \mathbf{F}(t)$ can also be written as a pair of parametric equations

$$x = f_1(t), \qquad y = f_2(t).$$

As t varies over the real numbers, the point $X(x, y)$ traces out a *parametric curve* in the plane. The vector valued function $\mathbf{F}(t)$ is called the *position vector* of the curve.

The line with the vector equation $\mathbf{X} = \mathbf{P} + t\mathbf{C}$ is a parametric curve with position vector $\mathbf{F}(t) = \mathbf{P} + t\mathbf{C}$ and components

$$f_1(t) = p_1 + tc_1, \qquad f_2(t) = p_2 + tc_2.$$

EXAMPLE 1 Find the vector equation for a particle which moves counterclockwise around the unit circle, and is at the point $(1, 0)$ at time $t = 0$, shown in Figure 10.6.1.

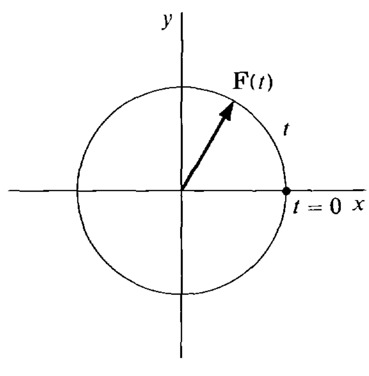


Figure 10.6.1

The motion is given by the parametric equations

$$x = \cos t, \qquad y = \sin t,$$

and the vector equation $\quad \mathbf{X} = \cos t\mathbf{i} + \sin t\mathbf{j}.$

EXAMPLE 2 A ball thrown at time $t = 0$ with initial velocity of v_1 in the x direction and v_2 in the y direction will follow the parabolic curve

$$x = v_1 t, \qquad y = v_2 t - 16t^2.$$

The curve (Figure 10.6.2) has the vector equation $\mathbf{X} = v_1 t\mathbf{i} + (v_2 t - 16t^2)\mathbf{j}$.

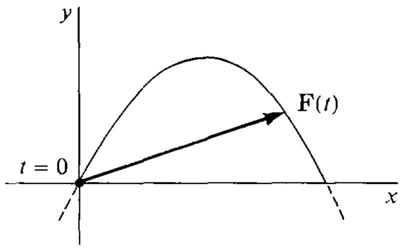


Figure 10.6.2

EXAMPLE 3 A point on the rim of a wheel rolling along a line traces out a curve called *a cycloid*. Find the vector equation for the cycloid if the wheel has radius one, rolls at one radian per second along the x-axis, and starts at $t = 0$ with the point at the origin.

As we can see from the close-up in Figure 10.6.3, the parametric equations are

$$x = t - \sin t, \qquad y = 1 - \cos t.$$

The vector equation is $\mathbf{X} = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$.

A vector valued function in three dimensions can be written in the form

$$\mathbf{F}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$$

and has the three components f_1, f_2, and f_3. The equation $\mathbf{X} = \mathbf{F}(t)$ can be written as three parametric equations

$$x = f_1(t), \qquad y = f_2(t), \qquad z = f_3(t),$$

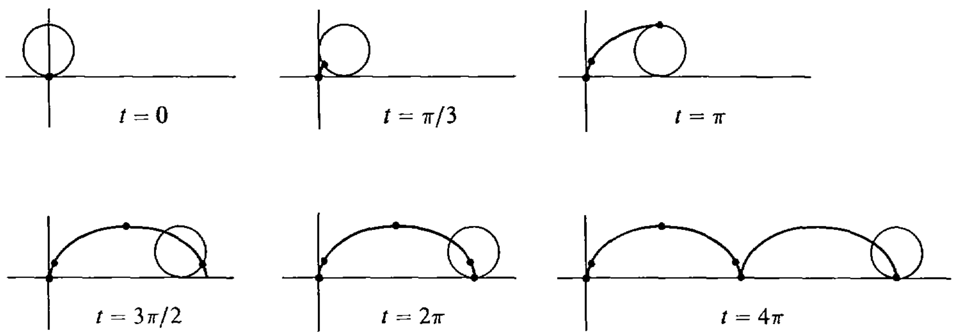
and as t varies over the reals we get a parametric curve in space.

EXAMPLE 4 The space curve

$$\mathbf{X} = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$$

is a *circular helix*. The point (x, y) goes around a horizontal circle of radius one whose center is rising vertically at a constant rate (Figure 10.6.4).

EXAMPLE 5 In economics the price vector may change with time and thus be a vector valued function of t. Find the price vector function $\mathbf{P}(t)$ for three



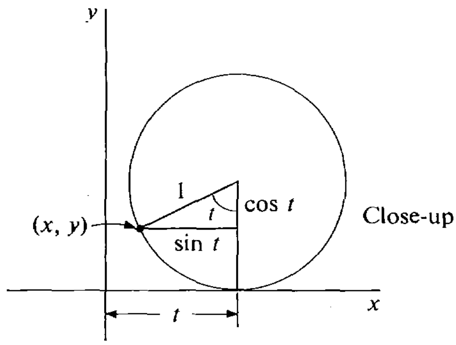


Figure 10.6.3

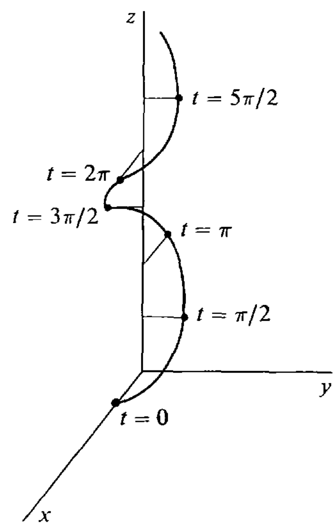


Figure 10.6.4

commodities such that the first commodity has price t^2, the second has price $t + 1$, and the price of the third commodity is the sum of the other two ($t \geq 0$). The answer is

$$\mathbf{P}(t) = t^2\mathbf{i} + (t + 1)\mathbf{j} + (t^2 + t + 1)\mathbf{k}.$$

PROBLEMS FOR SECTION 10.6

Find the vector equations for the motion of the given point in the plane. The positions at $t = 0$ and $t = 1$ are as shown in the figures.

1 A point moving along the parabola $y = x^2$ in such a way that $x = 3t$.

2 A point moving along $y = x^2$ so that $xy = t$.

3 A point moving upward along the line $y = 2x$ so that its distance from the origin at time t is t^3.

4 A wheel of radius one is turning at the rate of one radian per second. At the same time its center is moving along the x-axis at one unit per second. Find the motion of a point on the circumference of the wheel.

5 The point at distance one from the origin in the direction of the point $(t, 1)$.

6 The point where the parabola $y = x^2$ intersects the line through the origin which makes an angle t with the x-axis.

7 The point halfway between a point P going around the circle $x^2 + y^2 = 1$ at one radian per second and a point Q going around the same circle at 3 radians per second.

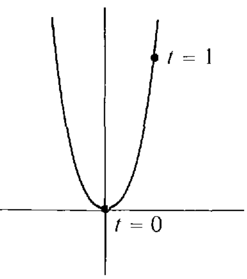
8 A wheel of radius one rolls along the x-axis at one radian per second. Find the motion of a point on the circumference of a concentric axle of radius $\frac{1}{2}$.

9 A circle of radius one rolls around the outside of the circle $x^2 + y^2 = 9$ at one radian per second. Find the motion of a point on the circumference of the smaller circle.

10 Find the motion of the point in Problem 9 if the small circle rolls around the inside of the large circle.

11 A string is unwound from a circular reel of radius one at one radian per second. The string is held taut and forms a line tangent to the reel. Find the motion of the end of the string.

1.



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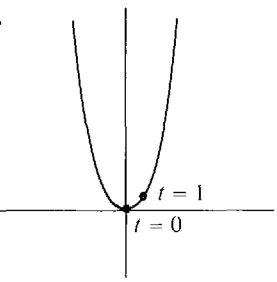

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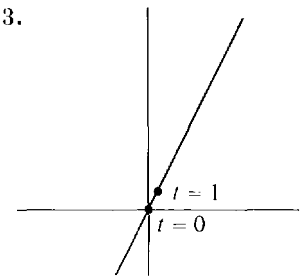

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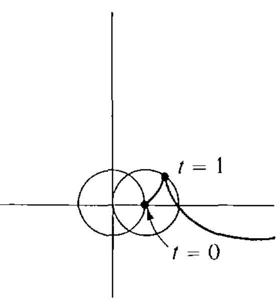

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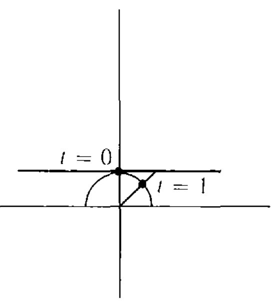

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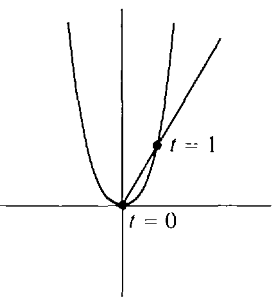

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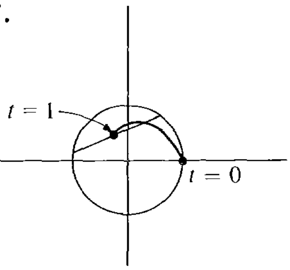

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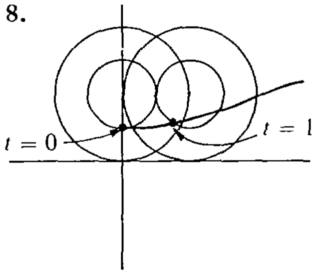

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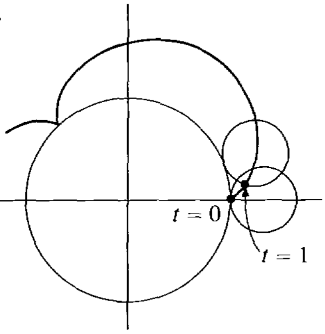

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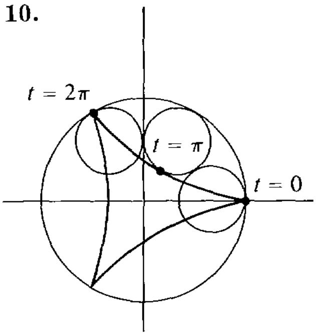

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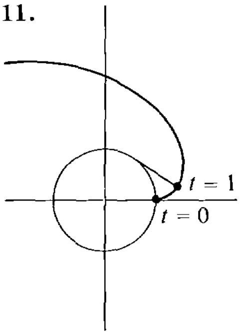


In Problems 12–23, find the vector equation for the motion of the given point in space.

12 A point moving so that at time t its position vector has length t^2 and direction cosines $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$.

13 A point X moving at one radian per second counterclockwise around a horizontal unit circle whose center is at $(0, 0, t^2)$ at time t. (At $t = 0$, $\mathbf{X} = \mathbf{i}$.)

14 The point which at time t is at distance one from the origin in the direction of the vector $t\mathbf{i} + \mathbf{j} + t^2\mathbf{k}$.

15 The point at distance one from the point $P(1, 2, 1)$ in the direction of the vector $t^2\mathbf{j} + (t^2 - 1)\mathbf{k}$.

16 The point where the line through the origin in the direction of $\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$ intersects the plane $x + 2y + 3z = 1$.

17 The point halfway between a point P going around the circle $x^2 + y^2 = 1$ in the (x, y) plane at one radian per second and a point Q going around the circle $x^2 + z^2 = 1$ in the (x, z) plane at 2 radians per second. (At $t = 0$, $\mathbf{P} = \mathbf{Q} = \mathbf{i}$. Both motions are counterclockwise.)

18 The point at distance $f(t)$ from the point $\mathbf{P}(t)$ in the direction of the vector $\mathbf{D}(t)$.

19 The point on the plane $x + y + z = 1$ which is nearest to the point

$$\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 6\mathbf{k}.$$

20 The point where the rotating plane $x \cos t + y \sin t = 0$ intersects the line through $(1, 1, 1)$ and $(2, 3, 4)$.

21 The point on the rotating plane $x \cos t + y \sin t = 0$ which is nearest to the point $t\mathbf{i} + 2t\mathbf{j} + 3t\mathbf{k}$.

22 Find the price vector $\mathbf{P}(t)$ for three commodities such that the first has price $1/t$, the second has double the price of the first, and the sum of the prices is 4 $(t \geq 1)$.

23 Find the price vector $\mathbf{P}(t)$ of three commodities such that the product of the three prices is one, the first commodity has price $2t$, and the third commodity has price $t + 1$ $(t \geq 1)$.

10.7 VECTOR DERIVATIVES

The derivative of a vector valued function is defined in terms of its components. We shall state the definitions for three dimensions. The two-dimensional case is similar.

DEFINITION

Given a vector valued function

$$\mathbf{F}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k},$$

the ***derivative*** $\mathbf{F}'(t)$ *is defined by*

$$\mathbf{F}'(t) = f_1'(t)\mathbf{i} + f_2'(t)\mathbf{j} + f_3'(t)\mathbf{k}.$$

$\mathbf{F}'(t)$ *exists if and only if* $f_1'(t)$, $f_2'(t)$, *and* $f_3'(t)$ *all exist.*

When we use the notation $\mathbf{X} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the derivative is written

$$\frac{d\mathbf{X}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}.$$

EXAMPLE 1 Find $d\mathbf{X}/dt$ where

$$\mathbf{X} = t^{1/3}\,\mathbf{i} + \frac{1}{t+1}\mathbf{j} + 2t\mathbf{k}, \qquad t \neq -1.$$

$$d\mathbf{X}/dt = \tfrac{1}{3}t^{-2/3}\,\mathbf{i} - (t+1)^{-2}\,\mathbf{j} + 2\mathbf{k}.$$

$d\mathbf{X}/dt$ is undefined at $t = 0$ and $t = -1$.

If $\mathbf{X}$ is the position vector of a line L, $\mathbf{X} = \mathbf{P} + t\mathbf{C}$, then the derivative of $\mathbf{X}$ is the constant direction vector $\mathbf{C}$, $d\mathbf{X}/dt = \mathbf{C}$. For

$$\frac{d\mathbf{X}}{dt} = \frac{d(p_1 + c_1t)}{dt}\mathbf{i} + \frac{d(p_2 + c_2t)}{dt}\mathbf{j} + \frac{d(p_3 + c_3t)}{dt}\mathbf{k}$$

$$= c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k} = \mathbf{C}.$$

The next two theorems show the geometric meaning of the vector derivative.

THEOREM 1

Given a curve $\mathbf{X} = \mathbf{F}(t)$ *in the plane, if* $\mathbf{F}'(t_0) \neq \mathbf{0}$ *then* $\mathbf{F}'(t_0)$ *is a direction vector of the line tangent to the curve at* t_0.

PROOF

Case 1 The curve is not vertical at t_0. The tangent line has slope

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{f_2'(t_0)}{f_1'(t_0)}$$

at t. Therefore the vector

$$\mathbf{F}'(t_0) = f_1'(t_0)\mathbf{i} + f_2'(t_0)\mathbf{j}$$

is a direction vector of the tangent line.

Case 2 The curve is vertical at t_0. Then $f'_1(t_0) = 0$, so $\mathbf{F}'(t_0) = f'_2(t_0)\mathbf{j}$ is a direction vector of the vertical tangent line. $\mathbf{F}'(t_0)$ is shown in Figure 10.7.1 for a curve $\mathbf{X} = \mathbf{F}(t)$.

For curves in space we can use the vector derivative to *define* the tangent line.

DEFINITION

If $\mathbf{X} = \mathbf{F}(t)$ *is a curve in space and* $\mathbf{F}'(t_0) \neq \mathbf{0}$, *the **tangent line** of the curve at* t_0 *is the line with position vector* $\mathbf{F}(t_0)$ *and direction vector* $\mathbf{F}'(t_0)$.

A vector parallel to $\mathbf{F}'(t_0)$ *is said to be a **tangent vector** of the curve at* t_0.

EXAMPLE 2 Find the vector equation of the tangent line for the spiral

$$\mathbf{F}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \tfrac{1}{4}t\mathbf{k}$$

at the point $t = \pi/3$.

The derivative is $\mathbf{F}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \tfrac{1}{4}\mathbf{k}$.

At $t = \pi/3$ the tangent line has the equation

$$\mathbf{X} = \mathbf{F}(\pi/3) + t\mathbf{F}'(\pi/3)$$

or

$$\mathbf{X} = \left(\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} + \frac{\pi}{12}\mathbf{k}\right) + t\left(-\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{1}{4}\mathbf{k}\right).$$

The tangent line is shown in Figure 10.7.2.

We have seen that the direction of the vector derivative is tangent to the curve. We next discuss the length of the vector derivative.

Suppose all the derivatives dx/dt, dy/dt, and dz/dt are continuous on an interval $a \leq t \leq b$. Recall that in two dimensions the length of the curve is defined as

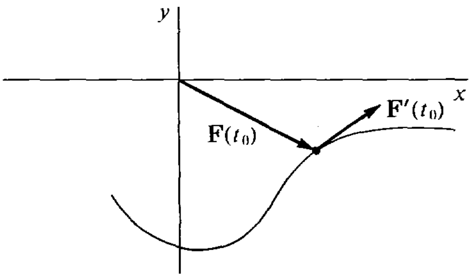


Figure 10.7.1

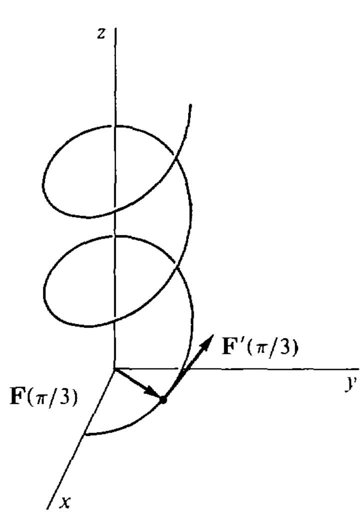


Figure 10.7.2

the integral

$$s = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt.$$

The *length of a curve in space* is defined in a similar way,

$$s = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}\, dt.$$

EXAMPLE 3 Find the length of the helix

$$\mathbf{X} = \cos t\mathbf{i} + \sin t\mathbf{j} + \tfrac{1}{4}t\mathbf{k},$$

from $t = a$ to $t = b$.

$$\frac{dx}{dt} = -\sin t, \qquad \frac{dy}{dt} = \cos t, \qquad \frac{dz}{dt} = \frac{1}{4}.$$

$$s = \int_a^b \sqrt{\sin^2 t + \cos^2 t + \tfrac{1}{16}}\, dt$$

$$= \int_a^b \sqrt{1 + \tfrac{1}{16}}\, dt = \int_a^b \frac{\sqrt{17}}{4}\, dt = \frac{\sqrt{17}}{4}(b - a).$$

THEOREM 2

Let $\mathbf{X} = \mathbf{F}(t)$ be a curve in space such that all the derivatives dx/dt, dy/dt, and dz/dt are continuous for $a \le t \le b$. Then the vector derivative $d\mathbf{X}/dt$ has length ds/dt, where s is the length of the curve from a to t. That is,

$$|d\mathbf{X}/dt| = ds/dt.$$

PROOF We have

$$s = \int_a^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt,$$

$$\frac{d\mathbf{X}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}.$$

Therefore

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \left|\frac{d\mathbf{X}}{dt}\right|.$$

If a particle moves in space so that its position vector at time t is $\mathbf{S} = \mathbf{F}(t)$, the vector derivative is called the *velocity vector*,

$$\mathbf{V} = \frac{d\mathbf{S}}{dt}.$$

The length of the velocity vector is called the *speed* of the particle. Theorems 1 and 2 show that:

The velocity vector **V** *is tangent to the curve.*

The speed |**V**| *is equal to the rate of change of the length of the curve,*

$$|\mathbf{V}| = \frac{ds}{dt}.$$

The second derivative is called the *acceleration vector*

$$\mathbf{A} = \frac{d\mathbf{V}}{dt} = \frac{d^2\mathbf{S}}{dt^2}.$$

EXAMPLE 4 Find the velocity, speed, and acceleration of a particle which moves around the unit circle with position vector

$$\mathbf{S} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}.$$

Velocity: $\mathbf{V} = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}.$

Speed: $|\mathbf{V}| = \sqrt{\sin^2 t + \cos^2 t} = 1.$

Acceleration: $\mathbf{A} = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}.$

As Figure 10.7.3 shows, the velocity **V** is tangent to the circle and the acceleration **A** points to the center of the circle.

EXAMPLE 5 Find the velocity, speed, and acceleration of a ball moving on the parabolic curve

$$S = v_1 t\,\mathbf{i} + (v_2 t - 16t^2)\mathbf{j}.$$

Velocity: $\mathbf{V} = v_1\mathbf{i} + (v_2 - 32t)\mathbf{j}.$

Speed: $|\mathbf{V}| = \sqrt{v_1^2 + (v_2 - 32t)^2}.$

Acceleration: $\mathbf{A} = -32\mathbf{j}.$

We see in Figure 10.7.4 that the velocity vector is tangent to the parabola, while the acceleration vector points straight down.

EXAMPLE 6 Find the position vector of a particle which moves with velocity

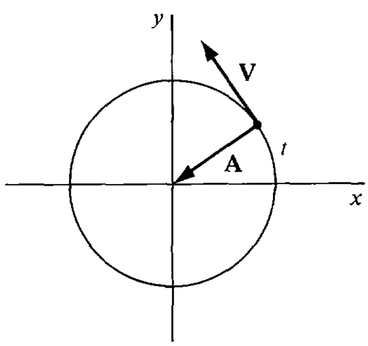


Figure 10.7.3

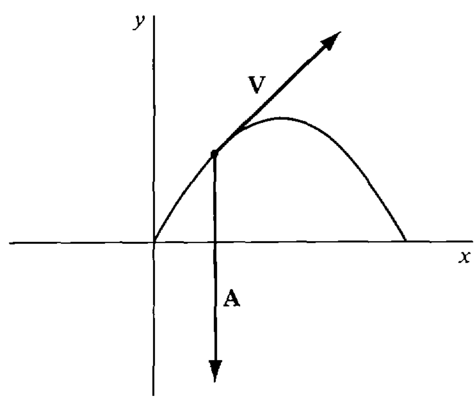


Figure 10.7.4

$$\mathbf{V} = -\sin t\mathbf{i} + \cos t\mathbf{j} + \sin t\cos t\mathbf{k}$$

and at time $t = 0$ has position $\mathbf{F}(0) = \mathbf{i} + 2\mathbf{k}$.

We find each component separately by integration.

$$f_1'(t) = -\sin t, \qquad f_1(0) = 1.$$
$$f_1(t) = \cos t + C_1.$$
$$1 = \cos 0 + C_1, \qquad C_1 = 0.$$
$$f_1(t) = \cos t.$$

$$f_2'(t) = \cos t, \qquad f_2(0) = 0.$$
$$f_2(t) = \sin t + C_2.$$
$$0 = \sin 0 + C_2, \qquad C_2 = 0.$$
$$f_2(t) = \sin t.$$

$$f_3'(t) = \sin t\cos t, \qquad f_3(0) = 2.$$
$$f_3(t) = \tfrac{1}{2}\sin^2 t + C_3.$$
$$2 = \tfrac{1}{2}\sin^2 0 + C_3, \qquad C_3 = 2.$$
$$f_3(t) = \tfrac{1}{2}\sin^2 t + 2.$$

$$\mathbf{F}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + (\tfrac{1}{2}\sin^2 t + 2)\mathbf{k}.$$

The path of the particle is shown in Figure 10.7.5.

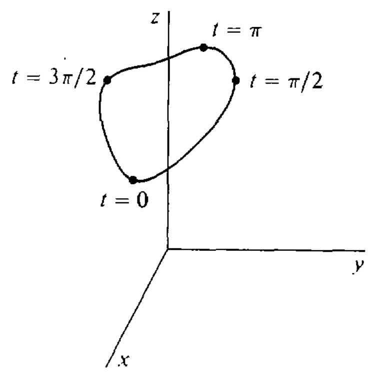


Figure 10.7.5

Let us briefly consider the derivative $\mathbf{P}'(t)$ of a price vector $\mathbf{P}(t)$. $\mathbf{P}'(t)$ is the *marginal price vector with respect to time.* It represents the rates at which the prices of all the commodities are changing. In a time of pure inflation, prices will increase but the ratios between prices of different commodities will stay the same, hence $\mathbf{P}'(t)$ will have the same direction as $\mathbf{P}(t)$. In a time of pure deflation $\mathbf{P}'(t)$ will have the opposite direction from $\mathbf{P}(t)$. Usually $\mathbf{P}'(t)$ is not parallel to $\mathbf{P}(t)$ at all, because the prices of some commodities are changing relative to others.

THEOREM 3 (Rules for Vector Derivatives)

Let $u = h(t)$ be a real function and let $\mathbf{X} = \mathbf{F}(t)$, $\mathbf{Y} = \mathbf{G}(t)$ be vector valued functions whose derivatives at t exist.

(i) Constant Rules $\dfrac{d(c\mathbf{X})}{dt} = c\dfrac{d\mathbf{X}}{dt}, \qquad \dfrac{d(\mathbf{C}u)}{dt} = \mathbf{C}\dfrac{du}{dt}.$

(ii) Sum Rule $\dfrac{d(\mathbf{X}+\mathbf{Y})}{dt} = \dfrac{d\mathbf{X}}{dt} + \dfrac{d\mathbf{Y}}{dt}.$

(iii) Inner Product Rule $\dfrac{d(\mathbf{X}\cdot\mathbf{Y})}{dt} = \mathbf{X}\cdot\dfrac{d\mathbf{Y}}{dt} + \dfrac{d\mathbf{X}}{dt}\cdot\mathbf{Y}.$

PROOF We prove (iii).

$$\mathbf{X}\cdot\mathbf{Y} = f_1(t)g_1(t) + f_2(t)g_2(t) + f_3(t)g_3(t).$$

$$\begin{aligned}\frac{d(\mathbf{X}\cdot\mathbf{Y})}{dt} &= f_1(t)g_1'(t) + f_2(t)g_2'(t) + f_3(t)g_3'(t)\\ &\quad + f_1'(t)g_1(t) + f_2'(t)g_2(t) + f_3'(t)g_3(t)\\ &= \mathbf{F}(t)\cdot\mathbf{G}'(t) + \mathbf{F}'(t)\cdot\mathbf{G}(t) = \mathbf{X}\cdot\frac{d\mathbf{Y}}{dt} + \frac{d\mathbf{X}}{dt}\cdot\mathbf{Y}.\end{aligned}$$

COROLLARY

Suppose $\mathbf{X} = \mathbf{F}(t)$ *is a curve whose distance* $|\mathbf{F}(t)|$ *from the origin is a constant* r_0. *Then the derivative* $\mathbf{F}'(t)$ *is perpendicular to* $\mathbf{F}(t)$ *whenever* $\mathbf{F}'(t) \neq \mathbf{0}$.

PROOF We use the Inner Product Rule. For all t,

$$\begin{aligned}r_0^2 &= \mathbf{F}(t)\cdot\mathbf{F}(t).\\ 0 = \frac{d(\mathbf{F}(t)\cdot\mathbf{F}(t))}{dt} &= \mathbf{F}(t)\cdot\mathbf{F}'(t) + \mathbf{F}'(t)\cdot\mathbf{F}(t)\\ &= 2\mathbf{F}(t)\cdot\mathbf{F}'(t).\end{aligned}$$

Therefore $\mathbf{F}(t)\cdot\mathbf{F}'(t) = 0$, so $\mathbf{F}(t) \perp \mathbf{F}'(t)$, as shown in Figure 10.7.6.

We see from the corollary that if a particle moves with constant speed $|\mathbf{V}| = v_0$, then its acceleration vector is always perpendicular to the velocity vector (Figure 10.7.7).

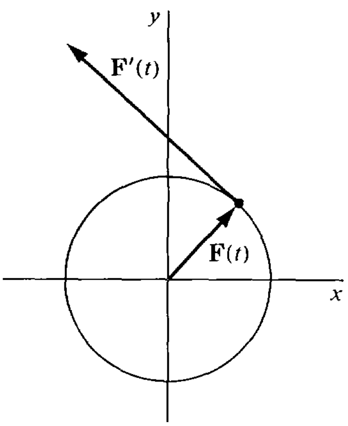


Figure 10.7.6

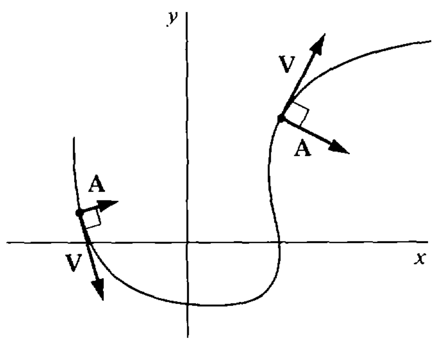


Figure 10.7.7 Motion with Constant Speed

PROBLEMS FOR SECTION 10.7

In Problems 1–9 find the derivative.

1 $\mathbf{X} = 5(\sin t\mathbf{i} + \cos t\mathbf{j})$

2 $\mathbf{X} = \cos(2t)\mathbf{i} + \sin(3t)\mathbf{j}$

3 $\mathbf{X} = \cos(e^t)\mathbf{i} + \sin(e^t)\mathbf{j}$

4 $\mathbf{X} = t^2\mathbf{i} + t^3\mathbf{j} - t\mathbf{k}$

5 $\mathbf{X} = -6(t\mathbf{i} - \ln t\mathbf{j} + e^t\mathbf{k})$

6 $\mathbf{X} = 4t^3(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$

7 $u = (\cos t\mathbf{i} + \sin t\mathbf{j}) \cdot (\sin t\mathbf{i} + \cos t\mathbf{j})$

8 $u = (2^t\mathbf{i} + 3^t\mathbf{j} + 4^t\mathbf{k}) \cdot (2t\mathbf{i} + 3t\mathbf{j} + 4t\mathbf{k})$

9 $u = |\cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}|$

10 Find the line tangent to the curve $\mathbf{X} = \sin^2 t\mathbf{i} + \cos^2 t\mathbf{j} + \sin t \cos t\mathbf{k}$ at $t = \pi/3$.

11 Find the line tangent to the curve $\mathbf{X} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ at the point $(1, 1, 1)$.

12 Find the line tangent to the cycloid $\mathbf{X} = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$ at $t = \pi/4$.

In Problems 13–25 find the velocity, speed, and acceleration.

13 $\mathbf{S} = 2t\mathbf{i} + 3t\mathbf{j} - 4t\mathbf{k}$

14 $\mathbf{S} = t^2\mathbf{i} + t\mathbf{j} + \mathbf{k}$

15 $\mathbf{S} = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$

16 The cycloid $\mathbf{S} = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$

17 $\mathbf{S} = \cos(e^t)\mathbf{i} + \sin(e^t)\mathbf{j}$

18 $\mathbf{S} = \cos t\mathbf{i} + \sin t\mathbf{j} + t^2\mathbf{k}$

19 $\mathbf{S} = (t^2 + 1)\mathbf{i} + (2t^2 + 1)\mathbf{j} + (-t^2 + 1)\mathbf{k}$

20 A point on the rim of a wheel of radius one in the (x, y) plane which is spinning counterclockwise at one radian per second and whose center at time t is at $(t, 0)$. (At $t = 0$, $\mathbf{S} = \mathbf{i}$.)

21 A bug which is crawling outward along a spoke of a wheel at one unit per second while the wheel is spinning at one radian per second. The center of the wheel is at the origin, and at $t = 0$, the bug is at the origin and the spoke is along the x-axis. (A spiral of Archimedes.)

22 The point at distance one from the origin in the direction of the vector

$$\mathbf{i} + t\mathbf{j} + \sqrt{2t}\mathbf{k}, \qquad t > 0.$$

23 A car going counterclockwise around a circular track $x^2 + y^2 = 1$ with speed $|2t|$ at time t. At $t = 0$ the car is at $(1, 0)$.

24 A point moving at speed one along the parabola $y = x^2$, going from left to right. ($\mathbf{S} = \mathbf{0}$ at $t = 0$.)

25 A point moving at speed y along the curve $y = e^x$ going from left to right. ($\mathbf{S} = \mathbf{j}$ at $t = 0$.)

In Problems 26–33 find the length of the given curve.

26 $\mathbf{X} = \cos t\mathbf{i} + \sin t\mathbf{j} + t^2\mathbf{k}, \quad 0 \le t \le 2$

27 $\mathbf{X} = \cos t\mathbf{i} + \frac{3}{5}\sin t\mathbf{j} - \frac{4}{5}\sin t\mathbf{k}, \quad 0 \le t \le 2\pi$

28 $\mathbf{X} = 6t\mathbf{i} - 8t\mathbf{j} + t^2\mathbf{k}, \quad 0 \le t \le 5$

29 $\mathbf{X} = 2t\mathbf{i} + 3t^2\mathbf{j} + 3t^3\mathbf{k}, \quad 0 \le t \le 1$

30 $\mathbf{X} = t\mathbf{i} + \dfrac{1}{\sqrt{2}}t^2\mathbf{j} + \dfrac{1}{3}t^3\mathbf{k}, \quad 0 \le t \le 1$

31 $\mathbf{X} = \cos^2 t\mathbf{i} + \sin^2 t\mathbf{j} + 2\sin t\mathbf{k}, \quad 0 \le t \le \pi$

32 $\mathbf{X} = \cosh t\mathbf{i} + \sinh t\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1$

33 $\mathbf{X} = \ln t\mathbf{i} + \sqrt{2}t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}, \quad 1 \le t \le 2$

In Problems 34–37 find the position vector of a particle with the given velocity vector and initial position.

34 $\mathbf{V} = e^t\mathbf{i} + e^{2t}\mathbf{j} + e^{3t}\mathbf{k}$, $\mathbf{F}(0) = \mathbf{0}$

35 $\mathbf{V} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $\mathbf{F}(1) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

36 $\mathbf{V} = \dfrac{\mathbf{i}}{t^2+1} + \dfrac{\mathbf{j}}{t^2-1} + \mathbf{k}$, $\mathbf{F}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

37 $\mathbf{V} = \dfrac{\mathbf{i}}{t-1} + \dfrac{\mathbf{j}}{t-2} + \dfrac{\mathbf{k}}{t-3}$, $\mathbf{F}(0) = \mathbf{0}$

38 Find the position vector of a particle whose acceleration vector at time t is $\mathbf{A} = \mathbf{i} + t\mathbf{j} + e^t\mathbf{k}$, if at $t = 0$ the velocity and position vectors are both zero.

39 Find the position vector $\mathbf{S}$ if $\mathbf{A} = \sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$, and at $t = 0$, $\mathbf{V} = \mathbf{0}$ and $\mathbf{S} = \mathbf{0}$.

□ 40 Show that if $\mathbf{U}$ is the unit vector of $\mathbf{X}$, then

$$\frac{d|\mathbf{X}|}{dt} = \frac{d\mathbf{X}}{dt}\cdot\mathbf{U}.$$

□ 41 Show using the Chain Rule that if $\mathbf{X}$ is the position vector of a curve and s is the length from 0 to t, then $d\mathbf{X}/ds$ is a unit vector tangent to the curve.

□ 42 Suppose a particle moves so that its speed is constant and its distance from the origin at time t is e^t. Show that the angle between the position and velocity vectors is constant.

□ 43 Prove that if $\mathbf{F}(t)$ is perpendicular to a constant vector $\mathbf{C}$ for all t, then $\mathbf{F}'(t)$ is also perpendicular to $\mathbf{C}$.

□ 44 Prove that if $\mathbf{F}(t)$ is parallel to a constant vector $\mathbf{C}$ for all t, then $\mathbf{F}'(t)$ is also parallel to $\mathbf{C}$.

□ 45 Prove the following differentiation rule for scalar multiples:

$$\frac{d(u\mathbf{X})}{dt} = u\frac{d\mathbf{X}}{dt} + \frac{du}{dt}\mathbf{X}.$$

□ 46 Prove the vector product rule $\dfrac{d(\mathbf{X}\times\mathbf{Y})}{dt} = \mathbf{X}\times\dfrac{d\mathbf{Y}}{dt} + \dfrac{d\mathbf{X}}{dt}\times\mathbf{Y}$.

10.8 HYPERREAL VECTORS

This section may be skipped without affecting the rest of the course. We introduce hyperreal vectors and use them to give an infinitesimal treatment of vector derivatives. We shall concentrate on three dimensions; the theory for two dimensions is similar.

A *hyperreal vector* in three dimensions is a vector

$$\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

whose components a_1, a_2, and a_3 are hyperreal numbers. The algebra of hyperreal vectors is in many ways similar to the algebra of hyperreal numbers. It begins with the notions of infinitesimal, finite, and infinite hyperreal vectors.

A hyperreal vector $\mathbf{A}$ is said to be *infinitesimal*, *finite*, or *infinite* if its length $|\mathbf{A}|$ is an infinitesimal, finite, or infinite number, respectively. Two hyperreal vectors $\mathbf{A}$ and $\mathbf{B}$ are said to be *infinitely close*, $\mathbf{A} \approx \mathbf{B}$, if their difference $\mathbf{B} - \mathbf{A}$ is infinitesimal (Figure 10.8.1).

EXAMPLE 1 Let ε be a positive infinitesimal and H be a positive infinite hyperreal

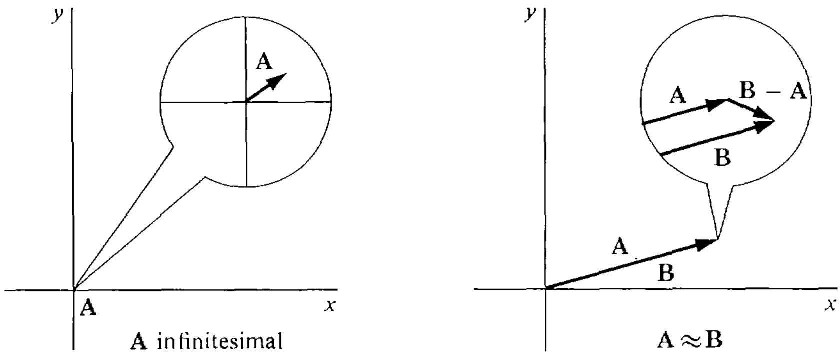


Figure 10.8.1

number. The vector $5\varepsilon\mathbf{i} + \varepsilon^2\mathbf{k}$ is infinitesimal. Its length is

$$\sqrt{25\varepsilon^2 + 0 + \varepsilon^4} = \varepsilon\sqrt{25 + \varepsilon^2} \approx 0.$$

The vector $\varepsilon\mathbf{i} + \mathbf{j} + \mathbf{k}$ is finite but not infinitesimal. Its length is

$$\sqrt{\varepsilon^2 + 1^2 + 1^2} = \sqrt{\varepsilon^2 + 2} \approx \sqrt{2}.$$

The vector $\mathbf{i} + \varepsilon\mathbf{j} + H\mathbf{k}$ is infinite. Its length is

$$\sqrt{1 + \varepsilon^2 + H^2} > H.$$

Our first theorem shows how these notions depend on the components of the vectors.

THEOREM 1

Let **A** *and* **B** *be hyperreal vectors.*

(i) **A** *is infinitesimal if and only if all of its components are infinitesimal.*

(ii) **A** *is finite if and only if all of its components are finite.*

(iii) **A** *is infinite if and only if at least one of its components is infinite.*

(iv) $\mathbf{A} \approx \mathbf{B}$ *if and only if* $a_1 \approx b_1$, $a_2 \approx b_2$, *and* $a_3 \approx b_3$.

PROOF (i), (ii), and (iii) are proved using the inequalities

$$|a_1| \le \sqrt{a_1^2 + a_2^2 + a_3^2}, \qquad |a_2| \le \sqrt{a_1^2 + a_2^2 + a_3^2}, \qquad |a_3| \le \sqrt{a_1^2 + a_2^2 + a_3^2},$$

$$\sqrt{a_1^2 + a_2^2 + a_3^2} \le |a_1| + |a_2| + |a_3|,$$

and (iv) follows easily from (i). We prove (i). Suppose **A** is infinitesimal. This means that its length

$$|\mathbf{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

is infinitesimal. The inequalities show that $|a_1|$, $|a_2|$, and $|a_3|$ are all between 0 and $|\mathbf{A}|$. Therefore all the components a_1, a_2, and a_3 are infinitesimal. On the other hand, if all the components are infinitesimal, then $|a_1| + |a_2| + |a_3|$ is infinitesimal, and by the last inequality, the length $|\mathbf{A}|$ is infinitesimal.

The following facts are obvious from the definitions.

The only infinitesimal real vector is $\mathbf{0}$.

Every real vector is finite.

Every infinitesimal vector is finite.

$\mathbf{A}$ *is infinitesimal if and only if* $\mathbf{A} \approx \mathbf{0}$.

Here is a list of algebraic rules for hyperreal vectors. Suppose the scalars and vectors ε, $\boldsymbol{\delta}$, are infinitesimal, c, $\mathbf{A}$, are finite but not infinitesimal, and H, $\mathbf{K}$ are infinite.

Negatives:

$-\boldsymbol{\delta}$ *is infinitesimal.*
$-\mathbf{A}$ *is finite but not infinitesimal.*
$-\mathbf{K}$ *is infinite.*

Sums:

$\boldsymbol{\delta}_1 + \boldsymbol{\delta}_2$ *is infinitesimal.*
$\mathbf{A} + \boldsymbol{\delta}$ *is finite but not infinitesimal.*
$\mathbf{A}_1 + \mathbf{A}_2$ *is finite (possibly infinitesimal).*
$\mathbf{K} + \boldsymbol{\delta}$ *and* $\mathbf{K} + \mathbf{A}$ *are infinite.*

Scalar multiples:

$\varepsilon\boldsymbol{\delta}$, $c\boldsymbol{\delta}$, and $\varepsilon\mathbf{A}$ *are infinitesimal.*
$c\mathbf{A}$ *is finite but not infinitesimal.*
$c\mathbf{K}$, $H\mathbf{A}$, *and* $H\mathbf{K}$ *are infinite.*

Inner products:

$\boldsymbol{\delta}_1 \cdot \boldsymbol{\delta}_2$ *and* $\boldsymbol{\delta} \cdot \mathbf{A}$ *are infinitesimal.*
$\mathbf{A}_1 \cdot \mathbf{A}_2$ *is finite (possibly infinitesimal).*

Each of these rules can be proved using Theorem 1. For example $\varepsilon\mathbf{A}$ is infinitesimal because each of its components εa_1, εa_2, and εa_3 is infinitesimal.

Other combinations, such as $\varepsilon\mathbf{K}$ and $H\boldsymbol{\delta}$, can be either infinitesimal, finite, or infinite.

As in the case of hyperreal numbers, our next step is to introduce the standard part. If $\mathbf{A}$ is a finite hyperreal vector, the *standard part* of $\mathbf{A}$ is the real vector

$$st(\mathbf{A}) = st(a_1)\mathbf{i} + st(a_2)\mathbf{j} + st(a_3)\mathbf{k}.$$

Since each component of $\mathbf{A}$ is infinitely close to its standard part, $\mathbf{A}$ is infinitely close to its standard part. Thus

st($\mathbf{A}$) *is the real vector infinitely close to* $\mathbf{A}$.

The standard part of an infinite hyperreal vector is undefined.

Here is a list of rules for standard parts of vectors. $\mathbf{A}$ and $\mathbf{B}$ are finite hyperreal vectors and c is a finite hyperreal number.

$$st(-\mathbf{A}) = -st(\mathbf{A})$$
$$st(\mathbf{A} + \mathbf{B}) = st(\mathbf{A}) + st(\mathbf{B})$$
$$st(c\mathbf{A}) = st(c)st(\mathbf{A})$$
$$st(\mathbf{A} \cdot \mathbf{B}) = st(\mathbf{A}) \cdot st(\mathbf{B})$$

$$st(\mathbf{A} \times \mathbf{B}) = st(\mathbf{A}) \times st(\mathbf{B})$$
$$st(|\mathbf{A}|) = |st(\mathbf{A})|$$

As an example we prove the equation for inner products,

$$\begin{aligned} st(\mathbf{A} \cdot \mathbf{B}) &= st(a_1b_1 + a_2b_2 + a_3b_3) \\ &= st(a_1)st(b_1) + st(a_2)st(b_2) + st(a_3)st(b_3) \\ &= st(\mathbf{A}) \cdot st(\mathbf{B}). \end{aligned}$$

Given a nonzero hyperreal vector $\mathbf{A}$, we may form its *unit vector* $\mathbf{U} = \mathbf{A}/|\mathbf{A}|$. The three components of $\mathbf{U}$ are the *direction cosines* of $\mathbf{A}$. As in the case of real vectors, $\mathbf{U}$ has length one and is parallel to $\mathbf{A}$.

Two new concepts which arise in the study of hyperreal vectors are vectors with real length and vectors with real direction. We say that $\mathbf{A}$ has *real length* if $|\mathbf{A}|$ is a real number. We say that $\mathbf{A}$ has *real direction* if the unit vector of $\mathbf{A}$ is real, or equivalently, the direction cosines of $\mathbf{A}$ are real.

There are four types of hyperreal vectors:

(a) Vectors with real length and real direction.
(b) Vectors with real length but nonreal direction.
(c) Vectors with nonreal length but real direction.
(d) Vectors with nonreal length and nonreal direction.

THEOREM 2

A vector is real if and only if it has both real length and real direction.

PROOF $\mathbf{A}$ has real length and direction if and only if $|\mathbf{A}|$ and $\mathbf{U} = \mathbf{A}/|\mathbf{A}|$ are both real if and only if $\mathbf{A} = |\mathbf{A}|\mathbf{U}$ is real.

EXAMPLE 2 Here are some vectors of type (b), (c), and (d), illustrated in Figure 10.8.2.

(b) The vector $\mathbf{B} = \sin\varepsilon\mathbf{i} + \cos\varepsilon\mathbf{j}$ has real length but nonreal direction (where ε is a positive infinitesimal). $\mathbf{B}$ has length one.

$$|\mathbf{B}| = \sqrt{\sin^2\varepsilon + \cos^2\varepsilon} = 1.$$

However, $\mathbf{B}$ is its own unit vector and is not real, so it has nonreal direction.

(c) The following vectors have nonreal lengths but real directions.

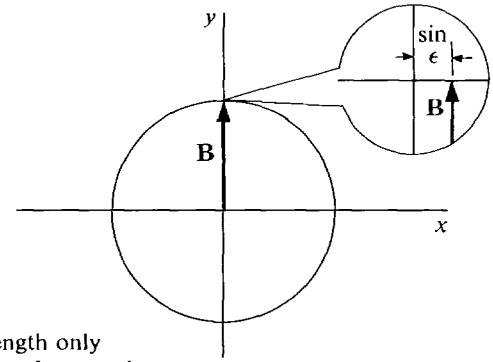
$3\varepsilon\mathbf{i} + 4\varepsilon\mathbf{j}$, infinitesimal length 5ε,
$(6 + 3\varepsilon)\mathbf{i} + (8 + 4\varepsilon)\mathbf{j}$, finite length $5(2 + \varepsilon)$,
$3H\mathbf{i} + 4H\mathbf{j}$, infinite length $5H$.

All three of these vectors are parallel and have the same real unit vector

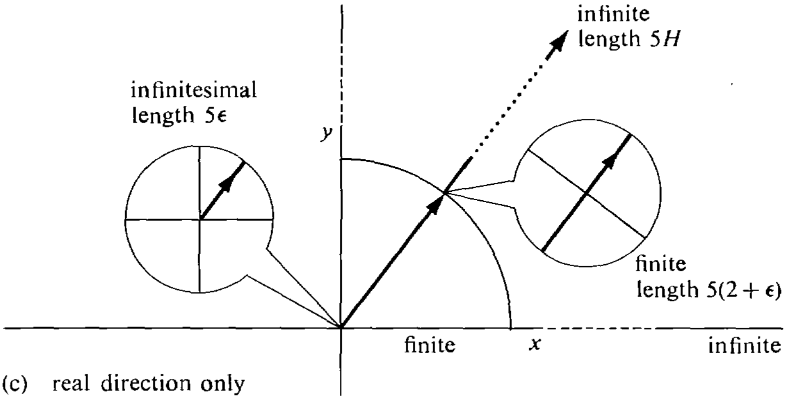
$$\mathbf{U} = \tfrac{3}{5}\mathbf{i} + \tfrac{4}{5}\mathbf{j}.$$

(d) The vector $\mathbf{D} = \mathbf{i} + \varepsilon\mathbf{j}$ has nonreal length and nonreal direction. Its length is $\sqrt{1 + \varepsilon^2}$, and its unit vector is

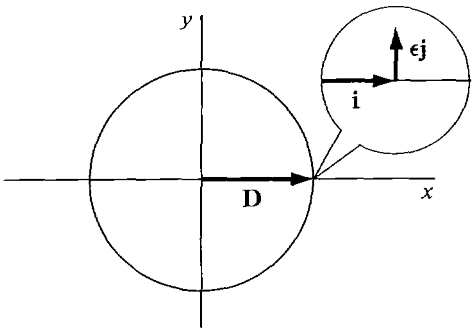
$$\mathbf{U} = \frac{1}{\sqrt{1 + \varepsilon^2}}\mathbf{i} + \frac{\varepsilon}{\sqrt{1 + \varepsilon^2}}\mathbf{j}.$$



(b) real length only
$\mathbf{B} = \sin \epsilon \mathbf{i} + \cos \epsilon \mathbf{j}$



(c) real direction only
$3\epsilon\mathbf{i} + 4\epsilon\mathbf{j}$
$3H\mathbf{i} + 4H\mathbf{j}$
$(6 + 3\epsilon)\mathbf{i} + (8 + 4\epsilon)\mathbf{j}$



(d) neither
$\mathbf{D} = \mathbf{i} + \epsilon\mathbf{j}$

Figure 10.8.2

Two hyperreal vectors **A** and **B** with unit vectors **U** and **V** are said to be *almost parallel* if either $\mathbf{U} \approx \mathbf{V}$ or $\mathbf{U} \approx -\mathbf{V}$.

EXAMPLE 3 The vectors

$$\mathbf{A} = 2\mathbf{i}, \qquad \mathbf{B} = 2\mathbf{i} + \varepsilon\mathbf{j}, \qquad \mathbf{C} = -\varepsilon\mathbf{i} + \varepsilon^2\mathbf{j}$$

are almost parallel to each other (Figure 10.8.3). Their unit vectors are

$$\mathbf{i}, \qquad \frac{2}{\sqrt{4+\varepsilon^2}}\mathbf{i} + \frac{\varepsilon}{\sqrt{4+\varepsilon^2}}\mathbf{j} \approx \mathbf{i}, \qquad \frac{-\varepsilon}{\sqrt{\varepsilon^2+\varepsilon^4}}\mathbf{i} + \frac{\varepsilon^2}{\sqrt{\varepsilon^2+\varepsilon^4}}\mathbf{j} \approx -\mathbf{i}.$$

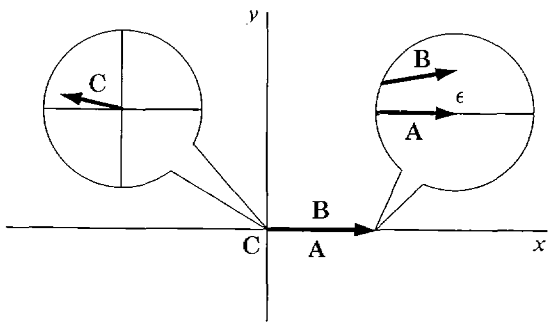


Figure 10.8.3

Let $\mathbf{A} \neq 0$ be a hyperreal vector with unit vector $\mathbf{U} = \mathbf{A}/|\mathbf{A}|$. **A** is almost parallel to the real unit vector $st(\mathbf{U})$. Thus every nonzero hyperreal vector is almost parallel to a real vector.

Now let us consider a vector valued function

$$\mathbf{F}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}.$$

Each of the real functions f_1, f_2, f_3 has a natural extension to a hyperreal function. Thus the real vector valued function **F** can be extended to a hyperreal vector valued function. When t is a hyperreal number, $\mathbf{F}(t)$ is defined if and only if all of $f_1(t)$, $f_2(t)$, and $f_3(t)$ are defined, and its value is

$$\mathbf{F}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}.$$

We shall now return to the study of vector derivatives.

THEOREM 3

The vector valued function $\mathbf{F}(t)$ *has derivative* **V** *at* t *if and only if*

$$\mathbf{V} = st\left(\frac{\mathbf{F}(t+\Delta t) - \mathbf{F}(t)}{\Delta t}\right)$$

for every nonzero infinitesimal Δt.

This theorem is exactly like the definition of the derivative of a real function in Chapter 2, except that it applies to a vector valued function.

PROOF OF THEOREM 3 Suppose first that $\mathbf{F}'(t) = \mathbf{V}$. This means that

$$f_1'(t)\mathbf{i} + f_2'(t)\mathbf{j} + f_3'(t)\mathbf{k} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

Then $$f_1'(t) = v_1, \qquad f_2'(t) = v_2, \qquad f_3'(t) = v_3.$$

Let Δt be a nonzero infinitesimal. Then

$$v_1 = st\left(\frac{f_1(t + \Delta t) - f_1(t)}{\Delta t}\right)$$

and similarly for v_2, v_3. It follows that

$$\mathbf{V} = st\left(\frac{\mathbf{F}(t + \Delta t) - \mathbf{F}(t)}{\Delta t}\right).$$

By reversing the steps we see that if the above equation holds for all nonzero infinitesimal Δt, then $\mathbf{V} = \mathbf{F}'(t)$.

We shall now discuss the increment and differential of a vector function. Given a curve

$$\mathbf{X} = \mathbf{F}(t),$$

t is a scalar independent variable and $\mathbf{X}$ a vector dependent variable. We introduce a new scalar independent variable Δt and a new vector dependent variable $\Delta\mathbf{X}$ with the equation

$$\Delta\mathbf{X} = \mathbf{F}(t + \Delta t) - \mathbf{F}(t).$$

$\Delta\mathbf{X}$ is called the *increment* of $\mathbf{X}$. $\Delta\mathbf{X}$ depends on both t and Δt, and is the vector from the point on the curve at t to the point on the curve at $t + \Delta t$ (Figure 10.8.4).

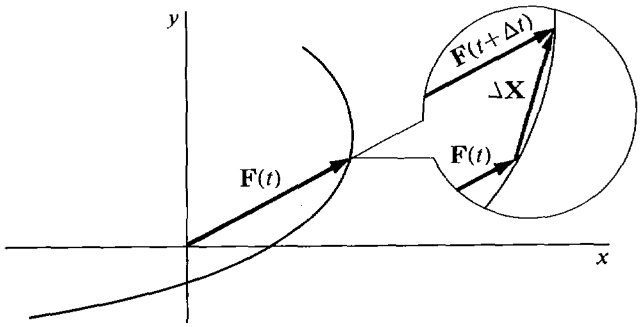


Figure 10.8.4 The Increment of **X**

Now suppose the vector derivative $\mathbf{F}'(t)$ exists. We introduce another vector dependent variable $d\mathbf{X}$ with the equation

$$d\mathbf{X} = \mathbf{F}'(t)\,\Delta t.$$

$d\mathbf{X}$ is called the *differential* of $\mathbf{X}$. It is customary to write dt for Δt, so we get the familiar quotient formulas

$$d\mathbf{X} = \mathbf{F}'(t)\,dt, \qquad \frac{d\mathbf{X}}{dt} = \mathbf{F}'(t).$$

The relationship between the vector increment and differential may be summarized as follows. At each value t where $\mathbf{F}'(t)$ exists and is not zero, and for each nonzero infinitesimal Δt, we have:

$d\mathbf{X}$ is an infinitesimal vector tangent to the curve $\mathbf{X} = \mathbf{F}(t)$.
$\Delta\mathbf{X}$ is an infinitesimal vector which is almost parallel to $d\mathbf{X}$.

$d\mathbf{X}$ and $\Delta\mathbf{X}$ are infinitely close compared to Δt, i.e,

$$\frac{\Delta\mathbf{X}}{\Delta t} \approx \frac{d\mathbf{X}}{dt}$$

as shown in Figure 10.8.5.

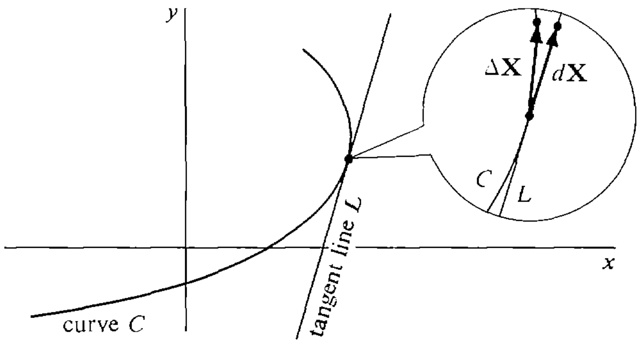


Figure 10.8.5

PROBLEMS FOR SECTION 10.8

In Problems 1–20, determine whether the given vector or scalar is infinitesimal, finite but not infinitesimal, or infinite. (ε, $\boldsymbol{\delta}$ are infinitesimal but not zero and H, $\mathbf{K}$ are infinite.)

1	$2\boldsymbol{\delta}_1 - 5\boldsymbol{\delta}_2$	**2**	$5\boldsymbol{\delta} - 3\mathbf{K}$
3	$H(2\mathbf{i} - \mathbf{j})$	**4**	$\varepsilon(5\mathbf{i} + \mathbf{j})$
5	$(2 + \varepsilon)\mathbf{i} + (3 - \varepsilon)\mathbf{j}$	**6**	$\dfrac{5\mathbf{i} + 6\mathbf{j} + \mathbf{k}}{\varepsilon}$
7	$\varepsilon\mathbf{i} - 4\mathbf{j} + H\mathbf{k}$	**8**	$\mathbf{K}/\lvert\mathbf{K}\rvert$
9	$\mathbf{K} \cdot \mathbf{K}$	**10**	$5\boldsymbol{\delta}/\lvert\boldsymbol{\delta}\rvert$
11	$(H_1\mathbf{i}) \cdot (H_2\mathbf{j})$	**12**	$(\mathbf{i} + \boldsymbol{\delta}_1) \cdot (\mathbf{j} + \boldsymbol{\delta}_2)$
13	$(H\mathbf{i} + \mathbf{j}) \cdot (\mathbf{i} + \varepsilon\mathbf{j})$		
14	$(\sqrt{H+1}\,\mathbf{i} + \sqrt{H}\mathbf{j}) \cdot (\sqrt{H+1}\,\mathbf{i} - \sqrt{H}\mathbf{j})$		
15	$(H\mathbf{i} + H^2\mathbf{j}) \cdot (H^{-2}\mathbf{i} + H^{-1}\mathbf{j})$	**16**	$\boldsymbol{\delta}_1 \times \boldsymbol{\delta}_2$
17	$(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \times \boldsymbol{\delta}$	**18**	$(\mathbf{i} + \boldsymbol{\delta}_1) \times (\mathbf{j} + \boldsymbol{\delta}_2)$
19	$(H_1\mathbf{i}) \times (H_2\mathbf{j})$	**20**	$(H\mathbf{i}) \times (\mathbf{j} + \boldsymbol{\delta})$

In Problems 21–30, compute the standard part. Assume $\mathbf{A}$, $\mathbf{B}$ are real.

21 $\dfrac{\cos(x + \Delta x)\mathbf{i} + \sin(x + \Delta x)\mathbf{j} - (\cos x\mathbf{i} + \sin x\mathbf{j})}{\Delta x}$

22 $(2 + \varepsilon)\mathbf{i} + (3 - \varepsilon)\mathbf{j}$

23 $(5 + 6\varepsilon)(2\mathbf{i} - 4\mathbf{j} + \mathbf{k})$

24 $\dfrac{2\varepsilon\mathbf{i} + 4\varepsilon^2\mathbf{j} + 6\varepsilon^3\mathbf{k}}{\varepsilon + \varepsilon^2 + \varepsilon^3}$

25 $\dfrac{(2H + 1)\mathbf{i} + (3H - 1)\mathbf{j} - H\mathbf{k}}{H + 4}$

26 $\dfrac{(\mathbf{A} + \boldsymbol{\delta})\cdot(\mathbf{B} + \boldsymbol{\delta}) - \mathbf{A}\cdot\mathbf{B}}{|\boldsymbol{\delta}|}$ where $st\left(\dfrac{\boldsymbol{\delta}}{|\boldsymbol{\delta}|}\right) = \mathbf{U}$

27 $|H\mathbf{i} + \mathbf{j}| - H$

28 $|H\mathbf{i} + \sqrt{H}\mathbf{j}| - H$

29 $\dfrac{|\mathbf{A} + \boldsymbol{\delta}| - |\mathbf{A}|}{|\boldsymbol{\delta}|}$ where $st\left(\dfrac{\boldsymbol{\delta}}{|\boldsymbol{\delta}|}\right) = \mathbf{U}$

30 $\dfrac{(x + \Delta x)^2(\mathbf{A} + \Delta x\mathbf{B}) - x^2\mathbf{A}}{\Delta x}$

In Problems 31–40 determine whether or not the vector has (a) real length, (b) real direction.

31 $H\mathbf{i} + \sqrt{H}\mathbf{j}$

32 $\mathbf{i} + \varepsilon\mathbf{j} + \varepsilon^2\mathbf{k}$

33 $(2\mathbf{i} + 2\sqrt{H}\mathbf{j} + H\mathbf{k})/(H + 2)$

34 $2H\mathbf{i} - 3H\mathbf{j}$

35 $\cos(2 + \varepsilon)\mathbf{i} + \sin(2 + \varepsilon)\mathbf{j}$

36 $\dfrac{\sqrt{5}}{\sqrt{1 + \varepsilon^2}}\mathbf{i} + \dfrac{\varepsilon}{\sqrt{1 + \varepsilon^2}}\mathbf{j} - \dfrac{2\varepsilon}{\sqrt{1 + \varepsilon^2}}\mathbf{k}$

37 $\dfrac{1}{\sqrt{1 + \varepsilon^2}}\mathbf{i} + \dfrac{\varepsilon}{\sqrt{1 + \varepsilon^2}}\mathbf{j} + \dfrac{\varepsilon}{\sqrt{1 + \varepsilon^2}}\mathbf{k}$

38 $\dfrac{\varepsilon}{\sqrt{1 + \varepsilon^2}}\mathbf{i} + \dfrac{2\varepsilon}{\sqrt{1 + \varepsilon^2}}\mathbf{j} - \dfrac{3\varepsilon}{\sqrt{1 + \varepsilon^2}}\mathbf{k}$

39 $5\cos(1 + \varepsilon)\mathbf{i} + 3\sin(1 + \varepsilon)\mathbf{j} + 4\sin(1 + \varepsilon)\mathbf{k}$

40 $5\cos(1 + \varepsilon)\mathbf{i} + 3\cos(1 + \varepsilon)\mathbf{j} + 4\cos(1 + \varepsilon)\mathbf{k}$

41 Prove that $st(\mathbf{A} + \mathbf{B}) = st(\mathbf{A}) + st(\mathbf{B})$.

42 Prove that $st(\mathbf{A} \times \mathbf{B}) = st(\mathbf{A}) \times st(\mathbf{B})$.

43 Prove that if **A** is infinite and **A** − **B** is finite, then **A** is almost parallel to **B**.

44 Prove that if **A** is finite but not infinitesimal and **A** − **B** is infinitesimal, then **A** is almost parallel to **B**.

45 Prove that a vector which is parallel to a real vector has a real direction.

The following problems use the notion of a continuous vector valued function. $\mathbf{F}(t)$ is said to be *continuous* at t_0 if each of the components $f_1(t)$, $f_2(t)$, and $f_3(t)$ is continuous at t_0.

□ 46 Prove that $\mathbf{F}(t)$ is continuous at t_0 if and only if whenever $t \approx t_0$, $\mathbf{F}(t) \approx \mathbf{F}(t_0)$.

□ 47 Assume $\mathbf{F}(t)$ and $\mathbf{G}(t)$ are continuous at t_0. Prove that the following functions are continuous at t_0.

$$\mathbf{F}(t) + \mathbf{G}(t), \qquad \mathbf{F}(t)\cdot\mathbf{G}(t), \qquad |\mathbf{F}(t)|, \qquad \mathbf{F}(t) \times \mathbf{G}(t).$$

□ 48 Prove that if $\mathbf{F}(t)$ and $h(t)$ are continuous at t_0, so is $h(t)\mathbf{F}(t)$.

EXTRA PROBLEMS FOR CHAPTER 10

1 Find the vector represented by the directed line segment $\overrightarrow{PQ}$ where $P = (4, 7)$, $Q = (9, -5)$.

2 Find the vector $\mathbf{A}/|\mathbf{B}|$ where $\mathbf{A} = 5\mathbf{i} - 10\mathbf{j}$, $\mathbf{B} = 3\mathbf{i} - 4\mathbf{j}$.

3 If $\mathbf{A} = 7\mathbf{i} + 2\mathbf{j}$, $\mathbf{B} = -4\mathbf{i} + \mathbf{j}$, find a vector $\mathbf{C}$ such that $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0}$.

4 An object originally has position vector $\mathbf{P} = 12\mathbf{i} - 5\mathbf{j}$ and is displaced twice, once by the vector $\mathbf{A} = 3\mathbf{i} + 3\mathbf{j}$ and once by the vector $\mathbf{B} = 6\mathbf{j}$. Find the new position vector.

5 Two traders initially have commodity vectors $\mathbf{A}_0 = 18\mathbf{i} + 2\mathbf{j}$, $\mathbf{B}_0 = 20\mathbf{j}$. They exchange in such a way that their new commodity vectors are equal, $\mathbf{A}_1 = \mathbf{B}_1$. Find their new commodity vectors.

6 Find a vector equation for the line through $P(2, 4)$ with direction vector $\mathbf{D} = \mathbf{i}$.

7 Find a vector equation for the line $3x + 4y = -1$.

8 Find the midpoint of the line AB where $A = (0, 0)$, $B = (-4, 2)$.

9 Find the point of intersection of the diagonals of the parallelogram $A(-1, -3)$, $B(0, -3)$, $C(5, 8)$, $D(4, 8)$.

10 Find the vector represented by $\overrightarrow{PQ}$ where $P = (4, 2, 1)$, $Q = (9, 6, 0)$.

11 Find the direction cosines of $\mathbf{A} = \mathbf{i} - 10\mathbf{j} + 2\mathbf{k}$.

12 If an object at rest has three forces acting on it and two of the forces are $\mathbf{F}_1 = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\mathbf{F}_2 = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, find the third force $\mathbf{F}_3$.

13 Find the force required to cause an object of mass 100 to accelerate with the acceleration vector $\mathbf{A} = \mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$.

14 If a trader has the commodity vector $\mathbf{A} = 5\mathbf{i} + 10\mathbf{j} + 15\mathbf{k}$ and sells the commodity vector $\mathbf{B} = 5\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}$, find his new commodity vector.

15 Find the vector equation of the line through $P(1, 4, 3)$ and $Q(1, 4, 4)$.

16 Find the vector equation of the line through $P(1, 1, 1)$ with direction cosines $(1/2, -1/2, 1/\sqrt{2})$.

17 Determine whether the vectors $\mathbf{A} = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$, $\mathbf{B} = 10\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$, are perpendicular.

18 Find the cost of the commodity vector $\mathbf{A} = 8\mathbf{i} + 20\mathbf{j} + 10\mathbf{k}$ at the price vector $\mathbf{P} = 6\mathbf{i} + 12\mathbf{j} + 15\mathbf{k}$.

19 Find the amount of work done by a force vector $\mathbf{F} = 10\mathbf{i} - 20\mathbf{j} + 5\mathbf{k}$ acting along the displacement vector $\mathbf{S} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$.

20 Find a vector in the plane perpendicular to $\mathbf{A} = -2\mathbf{i} + 3\mathbf{j}$.

21 Find a vector in space perpendicular to both

$$\mathbf{A} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}, \qquad \mathbf{B} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}.$$

22 Find two vectors in space perpendicular to each other and to $\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

23 Sketch the plane $x + 2y + 3z = 6$.

24 Sketch the plane $3x - z = 0$.

25 Find a scalar equation for the plane through the point $(1, 3, 2)$ with normal vector $\mathbf{N} = -\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

26 Find a scalar equation for the plane through the points $A(4, 1, 1)$, $B(2, 3, 4)$, $C(5, 1, 6)$.

27 Find the point where the line $\mathbf{X} = 2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} + t\mathbf{i}$ intersects the plane $x + y + z = 1$.

28 A bug is crawling along a spoke of a wheel towards the rim at a inches per second. At the same time the wheel is rotating counterclockwise at b radians per second. The center of the wheel is at $(0, 0)$ and at time $t = 0$, the bug is at $(0, 0)$. Find the vector equation for the motion of the bug, $0 \leq t \leq 1/a$.

29 The sphere $x^2 + y^2 + z^2 = 1$ is rotating about the z-axis counterclockwise at one radian per second. A bug crawls south at one inch per second along a great circle. At time $t = 0$ the bug is at $(0, 0, 1)$ and the great circle is in the (x, z) plane. Find the vector

equation for the motion of the bug, $0 \le t \le \pi$. (There are two possible answers.)

30 Find the velocity, speed, and acceleration of the bug in Problem 28.

31 Find the velocity, speed, and acceleration of the bug in Problem 29.

32 Find the derivative of $\mathbf{X} = (\cosh t)\mathbf{i} + (\sinh t)\mathbf{j}$.

33 Find the line tangent to the curve

$$\mathbf{X} = \frac{\mathbf{i}}{t+1} + \frac{\mathbf{j}}{t+2} + \frac{\mathbf{k}}{t+3} \quad \text{at } t = 0.$$

34 Find the length of the curve

$$\mathbf{X} = (\cosh^2 t)\mathbf{i} + (\sinh^2 t)\mathbf{j} + (\sqrt{8}\sinh t)\mathbf{k}, \qquad 0 \le t \le 1.$$

35 Find the position vector of a particle which moves with velocity

$$\mathbf{V} = (e^t \sin e^t)\mathbf{i} + (e^t \cos e^t)\mathbf{j} + e^t\mathbf{k},$$

if the particle is at the origin at $t = 0$.

36 If $\varepsilon > 0$ is infinitesimal, determine whether or not the vector $(\sin\varepsilon)\mathbf{i} + (1 - \cos\varepsilon)\mathbf{j}$ is infinitesimal.

37 Determine whether or not the vector in Problem 36 has real direction.

38 If $\varepsilon > 0$ is infinitesimal, find the standard part of the vector

$$\frac{(\sin\varepsilon)\mathbf{i} + \varepsilon^2\mathbf{j} + (e^\varepsilon - 1)\mathbf{k}}{\varepsilon}.$$

□ 39 Let $\mathbf{D}$ be a direction vector of a line L in the (x, y) plane. Prove that the set of all direction vectors of L is equal to the set of all scalar multiples of $\mathbf{D}$.

□ 40 Let $\mathbf{U}$ and $\mathbf{V}$ be perpendicular unit vectors in the plane. Prove that for any vector $\mathbf{A}$,

$$|\mathbf{A}|^2 = (\mathbf{A}\cdot\mathbf{U})^2 + (\mathbf{A}\cdot\mathbf{V})^2.$$

□ 41 Let $\mathbf{U}$ and $\mathbf{V}$ be perpendicular unit vectors in the plane. Prove that for any vector $\mathbf{A}$,

$$\mathbf{A} = (\mathbf{A}\cdot\mathbf{U})\mathbf{U} + (\mathbf{A}\cdot\mathbf{V})\mathbf{V}.$$

Hint: Let $\mathbf{B} = (\mathbf{A}\cdot\mathbf{U})\mathbf{U} + (\mathbf{A}\cdot\mathbf{V})\mathbf{V}$ and show that $\mathbf{B}\cdot\mathbf{U} = \mathbf{A}\cdot\mathbf{U}$ and $\mathbf{B}\cdot\mathbf{V} = \mathbf{A}\cdot\mathbf{V}$. $\mathbf{A}\cdot\mathbf{U}$ and $\mathbf{A}\cdot\mathbf{V}$ are called the $\mathbf{U}$ and $\mathbf{V}$ *components* of $\mathbf{A}$.

□ 42 Let $\mathbf{A}$ and $\mathbf{B}$ be two vectors in the plane which are not parallel. Prove that every vector $\mathbf{C}$ in the plane can be expressed uniquely in the form $\mathbf{C} = s\mathbf{A} + t\mathbf{B}$.

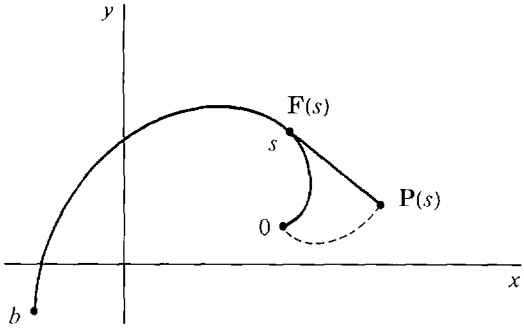
□ 43 Prove the Schwartz inequality $|\mathbf{A}\cdot\mathbf{B}| \le |\mathbf{A}||\mathbf{B}|$ for vectors $\mathbf{A}$, $\mathbf{B}$ in space.

□ 44 Prove that if s and t are positive scalars, then the angle between two vectors $\mathbf{A}$ and $\mathbf{B}$ in space is equal to the angle between $s\mathbf{A}$ and $t\mathbf{B}$.

□ 45 Let p be a plane in space with position vector $\mathbf{P}$ and nonparallel direction vectors $\mathbf{C}$ and $\mathbf{D}$. Prove that $\mathbf{Q}$ is a position vector of p if and only if $\mathbf{Q} = \mathbf{P} + s\mathbf{C} + t\mathbf{D}$ for some scalars s and t.
Hint: If $\mathbf{E}$ is a direction vector of p, then $\mathbf{E}\times\mathbf{D}$ is zero or parallel to $\mathbf{C}\times\mathbf{D}$, so $\mathbf{E}\times\mathbf{D} = s(\mathbf{C}\times\mathbf{D})$ for some s, $(\mathbf{E} - s\mathbf{C})\times\mathbf{D} = \mathbf{0}$, and hence $\mathbf{E} - s\mathbf{C}$ is parallel to $\mathbf{D}$.

□ 46 Let A, B, C be three distinct points in space whose plane does not pass through the origin. Prove that any vector $\mathbf{P}$ may be expressed uniquely in the form $\mathbf{P} = s\mathbf{A} + t\mathbf{B} + u\mathbf{C}$.
Hint: Consider the point where the line $\mathbf{X} = s\mathbf{A}$ intersects the plane with position vector $\mathbf{P}$ and direction vectors $\mathbf{B}$ and $\mathbf{C}$.

□ 47 Let C be a curve represented by the vector equation $\mathbf{X} = \mathbf{F}(s)$, $0 \le s \le b$. Assume that the length of the curve from $\mathbf{F}(0)$ to $\mathbf{F}(s)$ equals s, and that no tangent line crosses the curve. A string is stretched along the curve, attached at the end b, and carefully un-



wrapped starting at 0 as shown in the figure. Show that the point at the end of the string has the position vector $\mathbf{P}(s) = \mathbf{F}(s) + s\mathbf{F}'(s)$.

□ 48 A ball is thrown with initial velocity vector $\mathbf{V}_0 = b(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j})$ and position vector $\mathbf{S}_0 = \mathbf{0}$ at time $t = 0$. Its acceleration at time t is $\mathbf{A} = -32\mathbf{j}$. Find its position at time t, its maximum height, and the point where it hits the ground.