On the impact of indifferent voters on the likelihood of some voting paradoxes *

Vincent Merlin
CNRS & GEMMA
University of Caen
14032 Caen Cedex, France
merlin@econ.unicaen.fr

Fabrice Valognes
Dept.of Economics
The University of Namur
B-5000 Namur, Belgium
fabrice.valognes@fundp.ac.be

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Abstract

Most of the research to date on the probability that there is a Condorcet winner and on the likelihood that majority criterion and scoring rules agree has focused on the situation in which all voters have strict preference rankings on the candidates, with no indifference allowed. The purpose of the current paper is to consider the impact of voters' indifference on these studies. Attention is restricted to the case

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of three candidate elections in the limiting case of a large electorate. The probability that a voter will have a given preference on candidates is assumed to follow an extension of the widely used impartial culture condition. Using the same model, Gerhlein and Valognes computed the Condorcet efficiency of scoring rules, that is, the probability that a given scoring rule picks up the Condorcet winner, given that she exists. We complete their results with two studies. The first one considers the probability that a given scoring rule and the majority rule agree on a pair of candidates. The second one deals with the probability that the Condorcet winner is bottom ranked by a scoring rule.

1 Introduction

Most of the research to date on voting theory has focused on the situation in which all voters have complete preference rankings on the candidates, with no indifference allowed. Typically, a voting model assumes a set of voters N and a finite set of candidates A; each voter is able to rank without tie all the candidates according to her preference, i.e., her preference is a linear ordering on A[1]. Next, the question is to determine the best way to aggregate the individual preferences into a social ordering. However, many different criteria have been established to determine who the winner should be when more than two candidates are being considered. One criterion that has received a great deal of attention is attributed to Condorcet [5]: The winner should be the candidate that receives a majority of votes in all the pairwise comparisons. Such a candidate is called a *Condorcet winner*. Another approach dates back to Borda [4]: Each voter ranks without tie all the candidates, points are attributed to them according to their ranks in the preferences, and the candidate who obtains the highest total of points is declared as a winner. When the society has to choose among m candidates, the *Borda count* awards

[1] Recall, a linear ordering on A is a transitive and antisymmetric binary relation on A.

$m-1$ points for a first place, $m-2$ for a second, and so on down to 1 points for the next to the last, and 0 for the last ranked alternatives. But the Borda count is only one of the numerous *scoring rules* that can be designed by according points.

Almost all the studies on the voting rules, and in particular those that compared the Condorcet criterion and the scoring rules, make the assumption that individual preferences are linear orderings. In this line of research, one could quote many axiomatic results (see for example Young [27], Young and Levenglick [28]), most of the works that attempt to evaluate the likelihood of a Condorcet winner or the probability that a given scoring rule select the Condorcet winner (for exhaustive surveys on this literature, see Gehrlein [10], [11]), and even classic books such as the ones of Nurmi [19] and Saari [21].

Nevertheless, the pioneering work of Arrow [1] assumes that voters can be indifferent between two alternatives or more; in his genuine model, individual preferences are modeled by weak orderings, that is, binary relations that are transitive, complete, and reflexive. So, why is this model so often restricted to the case where preferences are linear orderings? A first reason is that the impossibility results in social choice literature can be equally stated for preference profiles of linear orderings or weak orderings. Secondly, while the number of linear orderings is easy to compute (for m candidates, there are $m!$), the number of weak orderings is obtained via a recursive formula. There are 13 possible weak preferences for 3 candidates, 71 for 4 candidates, 517 for 5, etc. Thus, taking into account weak orderings obviously complicates any characterization work or any computation of the likelihood of some given paradox. A third reason derives from the huge number of weak preferences: There are several ways to extend a voting rule defined on the set of strict preferences to the set of weak ordering. For example, Black [2] pointed out that there were different ways to define the Borda count for weak orderings. Smith [23] proposes an answer to this issue in his work on the characterization of scoring rules and suggests a way to extend their definition to weak orderings: when one voter is indifferent between candidates a and b and prefers them to candidate c (this preference is denoted by $a \sim b \succ c$), replace her by two "half voters", one which prefers a to b to c (this preference is denoted by $a \succ b \succ c$), and the other preferring b to a to c (this preference is denoted by $a \succ b \succ c$). His general argument is to replace a weak ordering by a "pool" of linear orderings. Tournament literature faces the similar issue of extending tournament solutions to weak tournaments [2]. For more on this subject, see Peris and Subiza [20] or Dutta and laslier [7]

All these reasons may explain why it is unusual to find in social choice literature, especially when the aim is to evaluate the qualities and flaws of a precise voting rule, articles that allow for indifferent voters.

Nevertheless, Fishburn and Gehrlein [8] computed the probability that the Condorcet winner exists when every voter picks her preference independently from the set of weak orderings. Their model has been used recently by Jones, Radcliff, Taber and Timpone [14], van Deemen [6] and Lepelley and Martin [16] to evaluate the likelihood of a Condorcet winner, and by Gerhlein and Valognes [13] to compute the probability that a given scoring rule selects the Condorcet winner.

The purpose of the current study is to consider the impact that voter indifference on candidates will have on the relationship between the majority rule and scoring rules. Attention is restricted to the case of three candidate elections in the limiting case of a large electorate. The probability that voters will have given preferences on candidates is assumed to follow an extension of the widely used impartial culture condition (IC). In this context, Gerhlein and Valognes [13] showed that taking into consideration indifferent voters could significantly increase the Condorcet efficiency of voting rules.

[2] A tournament is a complete and antisymetric binary relation on A. A weak tournament is a complete binary relation on A.

We intend to prove that similar conclusions can be derived for two other issues. The first one considers the probability that scoring rules and pairwise vote agree on pairs of candidates; this extends a previous work by Gehrlein and Fishburn [9], which did not take into account the case of indifferent voters. The second one deals with the probability that the Condorcet winner is bottom ranked by a scoring rule; here, the results with linear orders only have been provided by Tataru and Merlin [24] and Gerhlein and Lepelley [12].

The paper is organized as follows. In section 2, we present the voting model and the voting rules we shall examine. Next we introduce Fishburn and Gehrlein's Impartial Weak Order Culture condition (IWOC) we will use throughout the paper to evaluate the likelihood of paradoxes. In Section 3, we characterize the voting situations we want to evaluate the likelihood of and the main results. Section 4 is devoted to the proofs and Section 5 concludes the paper, proposing new issues to examine with the help of the IWOC assumption.

2 The Voting Model

2.1 Voters, Candidates and Preferences

We consider a population of voters, $N = \{1, \ldots, n\}$, who have to choose among candidates. In this paper, we shall restrict ourselves to the case of three candidates; $A = \{a, b, c\}$. There are six possible preference rankings on the candidates when indifference is not allowed:

$$\begin{array}{lll} a \succ b \succ c & n_1 & p_1 \\ a \succ c \succ b & n_2 & p_2 \\ b \succ a \succ c & n_3 & p_3 \\ b \succ c \succ a & n_4 & p_4 \\ c \succ a \succ b & n_5 & p_5 \\ c \succ b \succ a & n_6 & p_6 \end{array}$$

The number of individual with type i preference is denoted by n_i and p_i is the probability that a randomly selected voter has the associated preference ranking on candidates. For three candidates, there are 6 other preference types with partial indifference:

$$\begin{array}{lll} a \sim b \succ c & n_7 & p_7 \\ a \sim c \succ b & n_8 & p_8 \\ b \sim c \succ a & n_9 & p_9 \\ a \succ b \sim c & n_{10} & p_{10} \\ b \succ a \sim c & n_{11} & p_{11} \\ c \succ b \sim a & n_{12} & p_{12} \end{array}$$

The last possibility is a preference that represents complete indifference,

$$a \sim b \sim c \quad n_{13} \quad p_{13}$$

A voting situation $\tilde{n} = (n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9, n_{10}, n_{11}, n_{12}, n_{13})$ describes the distribution of the n voters on the different preference types.

2.2 Voting rules

The most famous criterion in voting theory is certainly the Condorcet criterion: It asserts that a candidate is a Condorcet winner whenever she is able to beat all his opponents in pairwise comparisons. For example a is a Condorcet winner whenever she beats b (inequality (1) is satisfied) and c (inequality (2) is satisfied).

$$n_1 + n_2 - n_3 - n_4 + n_5 - n_6 + n_8 - n_9 + n_{10} - n_{11} > 0 \quad (1)$$

$$n_1 + n_2 + n_3 - n_4 - n_5 - n_6 + n_7 - n_9 + n_{10} - n_{12} > 0 \quad (2)$$

Unfortunately, a Condorcet winner may not always exist and then another way to achieve a collective decision is needed. The class of scoring rules provide such alternative schemes. For three candidate elections, a scoring rule is characterized by the scoring vector $w_\lambda = (1, \lambda, 0)$: Each voter awards 1 point for her first choice, $\lambda \in [0, 1]$ for her second choice, and 0 for the last ranked candidate. A natural way to extend scoring rules for voters whose preference is partially indifferent among two or three candidates is the one suggested by Black [2] and Smith [23]: If a and b are tied, we should award them the average number of points they would have obtained if not tied. Thus, for the preference $a \sim b \succ c$, a and b get $(1 + \lambda)/2$ points and c zero point. Similarly, the preference $a \succ b \sim c$ is represented by the weights $(1, \lambda/2, \lambda/2)$, and the total indifference awards $(1 + \lambda)/3$ points to any candidate. Thus, candidate a is selected by the scoring rule w_λ for a voting situation $\tilde{n}$ iff :

$$(1 - \lambda)(n_1 - n_3) + n_2 - n_4 + \lambda(n_5 - n_6) + \frac{1 + \lambda}{2}(n_8 - n_9) + \frac{2 - \lambda}{2}(n_{10} - n_{11}) > 0 \quad (3)$$

$$n_1 - n_6 + (1 - \lambda)(n_2 - n_5) + \lambda(n_3 - n_4) + \frac{1 + \lambda}{2}(n_7 - n_9) + \frac{2 - \lambda}{2}(n_{10} - n_{12}) > 0 \quad (4)$$

The most famous scoring rules are the plurality rule, characterized by $w_0 = (1, 0, 0)$, the Borda count, $w_{\frac{1}{2}} = (1, \frac{1}{2}, 0)$ and the antiplurality rule, $w_1 = (1, 1, 0)$.

We can also use scoring vectors in a two stage voting process. A scoring runoff first ranks the candidates according to a scoring vector $(1, \lambda, 0)$, and next selects the winner among the top two on the basis of a pairwise comparison [3].

2.3 Probability Assumptions

When preferences are represented by linear orderings, the impartial culture condition (IC) assumes that $p_i = 1/(m!)$ for $i = 1, \ldots m!$, and that all voters determine their preference independently[4]. Fishburn and Gehrlein [8] suggested the following extension for possibly indifferent voters. Let k_1 denote the probability that the type of a randomly selected voter is a linear ordering, k_2 the probability that she is partially indifferent, and k_3 the probability that she is completely indifferent. Of course, $k_1 + k_2 + k_3 = 1$. $\mathbf{k} = (k_1, k_2, k_3)$. The impartial weak order culture condition (IWOC) then assumes that all the preference types within a class are equally likely to be observed. Thus, $p_i = k_1/6$ for $i = 1, \ldots 6$; $p_i = k_2/6$ for $i = 7, \ldots 12$ and $p_i = k_3$. When $k_2 = k_3 = 0$, we recover the classical impartial culture assumption, which has been widely used for the computation of many voting paradoxes.

The probability the Condorcet winner exists has been computed by Fishburn and Gehrlein [8] for a large number of voters:

$$P^{\infty}_{Con}(IWOC) = \frac{3}{4} + \frac{3}{2\pi} \arcsin\left(\frac{k_1 + k_2}{3k_1 + 2k_2}\right)$$

Under the same assumptions, Gehrlein and Valognes [13] have provided formulas that enable to evaluate the probability that a scoring rule picks the Condorcet winner, i.e., that inequalities (1) to (4) are simultaneously satisfied. They find out that this probability is maximized for the Borda count,

[3] In case of tie, we could assume that the first ranked candidate in the alphabetic order would win or go to the second stage. Nevertheless, for large populations, the probability of a tied outcome vanishes and can be neglected.

[4] For more on the IC condition and other assumptions that can be used when preferences are linear orderings, see Berg and Lepelley [3] and Gerhlein [11].

and that, for any scoring rule, it increases with k_2. We will now check whether such similar conclusions can be obtained for other voting situations.

3 The Likelihood of Paradoxes

3.1 Robustness of scoring rules over pairs of alternatives

The first event we evaluate is the probabilities that a scoring rule agree with the majority rule on a given pair of candidates. By doing so, the current study determine which scoring rule comes closer to satify Arrow's Independence of Irrelevant Alternatives (IIA) in his genuine framework, allowing for indifference.

Without loss of generality, we assume that the social ranking given by the scoring rule w_λ is $a \succ b \succ c$, the five other cases being similar. Then, we define by $P_{ab}^{\infty}(\lambda, k_1, k_2)$ (respectively by $P_{bc}^{\infty}(\lambda, k_1, k_2)$ and by $P_{ac}^{\infty}(\lambda, k_1, k_2)$) the conditional probability that the ranking on the pair $\{a, b\}$ (respectively on the pairs $\{b, c\}$ and $\{a, c\}$) agrees with the scoring ranking $a \succ b \succ c$ for the scoring vector λ, for the given values of k_1 and k_2 under the IWOC assumption, and for a large population (n goes to infinity). Note that $P_{ab}^{\infty}(\lambda, k_1, k_2)$ gives also the probability that a scoring rule w_λ and the scoring runoff method using the same scoring vector w_λ select the same winner.

Theorem 1. *The conditional probability that the weighted scoring rule using the scoring vector* $(1, \lambda, 0)$*, leads to the social ranking* $a \succ b \succ c$ *while the majority rule also ranks* a *before* b *under the IWOC assumption for a large* *electorate is given by:*

$$P_{ab}^{\infty}(\lambda, k1, k2) \;=\; \frac{1}{12} + \frac{1}{4\pi}\left[-\arcsin\left(\frac{4k1+3k2}{\sqrt{(3k1+2k2)(8k1z+k2z')}}\right)\right.$$
$$\left. +\;\arcsin\left(\frac{4k1+3k2}{2\sqrt{(3k1+2k2)(8k1z+k2z')}}\right)\right]$$

where $z = (1-\lambda+\lambda^2)$, $z' = (5-2\lambda+2\lambda^2)$, $\lambda \in [0,1]$, *and for* $k1+k2>0$.

Theorem 2. *The conditional probability that the weighted scoring rule using the scoring vector* $(1,\lambda,0)$*, leads to the social ranking* $a \succ b \succ c$ *while the majority rule also ranks* b *before* c *under the IWOC assumption for a large electorate is the same as the probability that the weighted scoring rule using the scoring vector* $(1,\lambda,0)$*, leads to the social ranking* $a \succ b \succ c$ *while the majority rule also ranks* a *before* b*.*

$$P_{ab}^{\infty}(\lambda, k1, k2) \;=\; P_{bc}^{\infty}(\lambda, k1, k2)$$

Theorem 3. *The conditional probability that the weighted scoring rule using the scoring vector* $(1,\lambda,0)$*, leads to the social ranking* $a \succ b \succ c$ *while the majority rule also ranks* a *before* c *under the IWOC assumption for a large electorate is given by:*

$$P_{ac}^{\infty}(\lambda, k1, k2) \;=\; 2 - \tfrac{3}{\pi}\arccos\left(\tfrac{4\,k1+3\,k2}{2\sqrt{3\,k1+2\,k2}\sqrt{8\,k1z+k2z'}}\right)$$

where $z = (1-\lambda+\lambda^2)$, $z' = (5-2\lambda+2\lambda^2)$, $\lambda \in [0,1]$, *and for* $k1+k2>0$.

These expressions are symmetric in λ around $\lambda = 1/2$ and decreases in z and z' so that each $P_{-}^{\infty}(\lambda, k1, k2)$ is maximized uniquely at $\lambda = 1/2$ and is minimized only by $\lambda \in \{0,1\}$. Thus the Borda score vector maximizes $P_{-}^{\infty}(\lambda, k1, k2)$, and the plurality and the reverse-plurality minimize $P_{-}^{\infty}(\lambda, k1, k2)$. This confirms the results obtained by van Newenhizen on the optimality of the Borda count regarding to majority criteria for probability distributions similar to the ones described by the IC assumption. For the special case $k1 = 1$ ($k2 = 0$), this reduces to the original case of IC, and to results due to Gehrlein and Fishburn [9]. Numerical evaluations are displayed in Table 1 and Table 2.

3.2 Ranking a Condorcet winner last

In this section we are interested in the probability $P^{\infty}_{CL}(\lambda, k1, k2)$ that a Condorcet winner is bottom ranked by any scoring method $(1, \lambda, 0)$, for $\lambda \in [0, 1]$ under the IWOC assumption for a large population. For candidate a, these situations are characterized by inequalities (1) and (2), and the fact that conditions (3) and (4) are not satisfied. We also denote $P^{\infty}_{CW}(\lambda, k1, k2)$ the probability that a scoring rule $w_\lambda = (1, \lambda, 0)$ select the Condorcet winner in the same conditions.

Theorem 4. *The conditional probability that the weighted scoring rule using the scoring vector* $(1, \lambda, 0)$ *elects the Condorcet loser when indifference is allowed is given by the following expression*

$$P^{\infty}_{CL}(\lambda, k1, k2) \quad = \quad P^{\infty}_{CW}(\lambda, k1, k2) - \left(\frac{\arcsin(\rho) + \arcsin(\rho/2)}{\pi/2 + \arcsin(\frac{k1+k2}{3k1+2k2})} \right)$$

with $\rho = \frac{4\,k1 + 3\,k2}{\sqrt{[8k1\,(1-\lambda+\lambda^2) + k2\,(5-2\lambda+2\lambda^2)](3k1+2k2)}}$

4 Proofs

The most natural way to compute likelihood of some events under the IWOC assumption is to follow the techniques proposed by Fishburn and Gehrlein [8] and also used by Gehrlein and Valognes [13]. Another possibility is to rely upon the techniques that have been proposed by Saari and Tataru [25]. In their paper, they estimate the likelihood of obtaining different social rankings as we modify the scoring rule. Tataru and Merlin [24] and Merlin, Tataru and Valognes [18] extended the range of application of theses techniques by using them for the computation of other events.

The results we present in this paper have been double checked with both techniques. By proving Theorem 1, we present a way to adapt the Saari-Tataru method for the IWOC case. The proof of Theorem 2 is similar, and is ommited. But, we will use the Saari-Tataru technique to prove Theorem 3. Unfortunately, this method requires heavy computations for the proof of Theorem 4; thus, we chose to present a more classical proof of this result, in the line of the Gehrlein-Fishburn papers.

4.1 Proofs of Theorem 1

4.1.1 The classical proof

The first step of the proof is to define three discrete variables x_1, x_2, and x_3. Each variable is based on a linear order on the set of alternatives that is randomly chosen by a voter according to a probability distribution p. The three random variables are

$$sc(a) > sc(b) : \begin{cases} x_1 & = & (1-\lambda) & p_1 - p_3 \\ & = & 1 & p_2 - p_4 \\ & = & \lambda & p_5 - p_6 \\ & = & 0 & p_7 + p_{12} + p_{13} \\ & = & (1+\lambda)/2 & p_8 - p_9 \\ & = & 1 - \lambda/2 & p_{10} - p_{11} \end{cases}$$

$$sc(b) > sc(c) : \begin{cases} x_2 & = & \lambda & p_1 - p_2 \\ & = & 1 & p_3 - p_5 \\ & = & (1-\lambda) & p_4 - p_6 \\ & = & 0 & p_9 + p_{10} + p_{13} \\ & = & (1+\lambda)/2 & p_7 - p_8 \\ & = & 1 - \lambda/2 & p_{11} - p_{12} \end{cases}$$

$$aMb : \begin{cases} x_3 & = & +1 & p_1 + p_2 + p_5 + p_8 + p_{10} \\ & = & -1 & p_3 + p_4 + p_6 + p_9 + p_{11} \\ & = & 0 & p_7 + p_{12} + p_{13} \end{cases}$$

Variable x_1 equals the difference in points awarded by a voter to a and b under indifference with the scores w_λ, and x_2 equals the difference in points between b and c under w_λ. Variable x_3 relates to the fact that we focus on a being the majority winner over b.

In going to the limit we use the multivariate extension of the central limit theorem saying that as $n \to \infty$ the limiting proportion in question is equal to a positive constant time the probability that $(\sqrt{n})$(average value over the orders in the profile of the variable)≥ 0 for each variable in the limit under the m-variate normal distribution with zero mean vector and correlation matrix derived from the average values of the pairwise products of the variables over the orders on the set of alternatives. Then, the three variate extension of the Central Limit Theorem [26] says that $(\bar{x}_1 n^{\frac{1}{2}}, \bar{x}_2 n^{\frac{1}{2}}, \bar{x}_3 n^{\frac{1}{2}})$ has a trivariate normal distribution with $E(x_j) = 0$ for each $j = 1, 2, 3$ and a covariance matrix V that is given by

$$V = \begin{bmatrix} 2k1z/3 + k2z'/12 & -k1z/3 - k2z'/24 & 2k1/3 + k2/2 \\ - & 2k1z/3 + k2z'/12 & -k1/3 - k2/4 \\ - & - & k1 + 2k2/3 \end{bmatrix}$$

where $z = (1 - \lambda + \lambda^2)$ and $z' = (5 - 2\lambda + 2\lambda^2)$, provided that V is positive definite. Since $det(V) > 0$ for all $\lambda \in [0, 1]$, the probability is given by the trivariate normal orthant probability with correlation matrix R given by

$$V = \begin{bmatrix} 1 & -1/2 & \frac{4k1+3k2}{\sqrt{\gamma(3k1+2k2)}} \\ - & 1 & -\frac{4k1+3k2}{2\sqrt{\gamma(3k1+2k2)}} \\ - & - & 1 \end{bmatrix}$$

where

$$\gamma = (8k1(1 - \lambda + \lambda^2) + k2(5 - 2\lambda + 2\lambda^2)).$$

Hence, $P_{ab}^{\infty}(\lambda, k_1, k_2) = 6\Phi_3(R_1)$. The trivariate extension of Shepard's Theorem of Median Dichotomy gives:

$$\Phi_3(R_1) = \frac{1}{12} + \frac{1}{4\pi}\left(\arcsin\left(\frac{4k1 + 3k2}{\sqrt{\gamma(3k1 + 2k2)}}\right) - \arcsin\left(\frac{4k1 + 3k2}{\sqrt{\gamma(3k1 + 2k2)}}\right)\right)$$

4.1.2 A similar problem

Consider now the following problem: each voter has a probability 1/13 to pick any of the 13 weak orderings, but the scores given for a strict orderings are multiplied by $\sqrt{k1}$, while the scores awarded to an alternatives for preferences 7 to 12 are multiplied by $\sqrt{k2}$. In a similar way, we want to evaluate the probability that the ranking given by the scoring rule λ is $a \succ b \succ c$ while a gets a majority of votes over b. The three random variables are:

$$sc(a) > sc(b) : \begin{cases} x_1' & = & (1-\lambda)\sqrt{k1} & p_1 - p_3 \\ & = & \sqrt{k1} & p_2 - p_4 \\ & = & \lambda\sqrt{k1} & p_5 - p_6 \\ & = & 0 & p_7 + p_{12} + p_{13} \\ & = & (1+\lambda)\sqrt{k2}/2 & p_8 - p_9 \\ & = & (1-\lambda/2)\sqrt{k2} & p_{10} - p_{11} \end{cases}$$

$$sc(b) > sc(c) : \begin{cases} x_2' & = & \lambda\sqrt{k1} & p_1 - p_2 \\ & = & \sqrt{k1} & p_3 - p_5 \\ & = & (1-\lambda)\sqrt{k1} & p_4 - p_6 \\ & = & 0 & p_9 + p_{10} + p_{13} \\ & = & (1+\lambda)\sqrt{k2}/2 & p_7 - p_8 \\ & = & (1-\lambda/2)\sqrt{k2} & p_{11} - p_{12} \end{cases}$$

$$aMb : \begin{cases} x'_3 & = & \sqrt{k1} & p_1 + p_2 + p_5 \\ & = & \sqrt{k2} & p_8 + p_{10} \\ & = & -\sqrt{k1} & p_3 + p_4 + p_6 \\ & = & -\sqrt{k2} & p_9 + p_{11} \\ & = & 0 & p_7 + p_{12} + p_{13} \end{cases}$$

By applying the same argument as in proof of Theorem 1, the three-variate extension of the Central Limit Theorem [26] says that $(\bar{x}'_1 n^{\frac{1}{2}}, \bar{x}'_2 n^{\frac{1}{2}}, \bar{x}'_3 n^{\frac{1}{2}})$. Thus, assuming $k1 = k2 = 6/13$ and modifying the weights in an appropriate way in the equations (1) to (4) lead to a similar problem as the one described by Theorem 1.

4.2 Proof of Theorem 2

In order to give the details of the computation of $P^{\infty}_{ac}(\lambda, k_1, k_2)$, we here choose to modify the problem stated in Theorem 2, by multiplying all the coefficient of the n_t's by $\sqrt{k_1}$ if $t = 1, \ldots, 6$ and by $\sqrt{k_2}$ if $t = 7, \ldots, 12$.

$$\sqrt{k1}(n_1 + n_2 + n_3 - n_4 - n_5 - n_6) + \sqrt{k2}(n_7 - n_9 + n_{10} - n_{12}) > 0 \quad (5)$$
$$\sqrt{k1}((1-\lambda)(n_1 - n_3) + n_2 - n_4 + \lambda(n_5 - n_6)) + \sqrt{k2}(\frac{(1+\lambda)}{2}(n_8 - n_9) + \frac{2-\lambda}{2}(n_{10} - n_{11})) > 0 \quad (6)$$
$$\sqrt{k1}(n_1 - n_6 + (1-\lambda)(n_2 - n_5) + \lambda(n_3 - n_4)) + \sqrt{k2}(\frac{1+\lambda}{2}(n_7 - n_9) + \frac{2-\lambda}{2}(n_{10} - n_{12})) > 0 \quad (7)$$

Following the arguments given by Saari and Tataru [25], Tataru and Merlin [17] and Merlin, Tataru and Valognes [18], the probability that these 3 inequalities are met simultaneously for a voting situation when $p_i = \frac{1}{13}$, $i = 1, \ldots, 13$ under the IWOC assumption for n large is equal to the surface of the spherical simplex T described by equation (5),(6),(7) on the surface of the unit sphere in $\mathbb{R}^3$, divided by the surface of this sphere. Let W_t be a normal vector for hyperplane T_t, described by equations (t), $t = 5, 6, 7$.

$$\begin{aligned} W5 &= (\sqrt{k1}, \sqrt{k1}, \sqrt{k1}, -\sqrt{k1}, -\sqrt{k1}, -\sqrt{k1}, 0, \sqrt{k2}, -\sqrt{k2}, \sqrt{k2}, -\sqrt{k2}, 0, 0) \\ W6 &= ((2-2a)\sqrt{k1}, 2\sqrt{k1}, (2a-2)\sqrt{k1}, -2\sqrt{k1}, 2a\sqrt{k1}, -2a\sqrt{k1}, 0, \sqrt{k2}(1+a), \sqrt{k2}(-1-a), \sqrt{k2}(2-a), \sqrt{k2}(a-2), 0, 0) \\ W7 &= (2a\sqrt{k1}, -2a\sqrt{k1}, 2\sqrt{k1}, (2-2a)\sqrt{k1}, -2\sqrt{k1}, (2a-2)\,l, \sqrt{k2}(1+a), \sqrt{k2}(-1-a), 0, 0, \sqrt{k2}(2-a), \sqrt{k2}(a-2), 0) \end{aligned}$$

The three hyperplanes T_5, T_6 and T_7 define a spherical simplex (a triangle here) on the surface of the unit sphere in $\mathbb{R}^3$. Let α_{tu} be the angle between the hyperplanes T_t and T_u. By the Gauss-Bonnet theorem, the surface S of the triangle is:

$$S = \alpha_{56} + \alpha_{57} + \alpha_{67} - \pi$$

$$\begin{aligned}
\alpha_{56} &= \arccos\left(-\frac{W5.W6}{||W5||.||W6||}\right) \\
&= \pi - \arccos(1/2\,\frac{4\,k1+3\,k2}{\sqrt{3\,k1+2\,k2}\sqrt{8\,\lambda^2 k1+8\,k1-8\,k1\lambda+5\,k2-2\,k2\lambda+2\,k2\lambda^2}}) \\
&= \pi - \arccos(1/2\,\frac{4\,k1+3\,k2}{\sqrt{3\,k1+2\,k2}\sqrt{8k1z+k2z'}})
\end{aligned}$$

$$\begin{aligned}
\alpha_{57} &= \arccos\left(-\frac{W5.W7}{||W5||.||W7||}\right) \\
&= \pi - \arccos(1/2\,\frac{4\,k1+3\,k2}{\sqrt{3\,k1+2\,k2}\sqrt{8\,\lambda^2 k1+8\,k1-8\,k1\lambda+5\,k2-2\,k2\lambda+2\,k2\lambda^2}}) \\
&= \pi - \arccos(1/2\,\frac{4\,k1+3\,k2}{\sqrt{3\,k1+2\,k2}\sqrt{8\,k1z+k2z'}})
\end{aligned}$$

where $z = (1 - \lambda + \lambda^2)$, $z' = (5 - 2\lambda + 2\lambda^2)$, and $\lambda \in [0, 1]$.

$$\begin{aligned}
\alpha_{67} &= \arccos\left(-\frac{W6.W7}{||W6||.||W7||}\right) \\
&= \frac{\pi}{3}
\end{aligned}$$

By dividing S by 4π, the surface of the sphere, and by multiplying it by 6, we obtain:

$$P^{\infty}_{ac}(\lambda, k1, k2) \;=\; 2 - \frac{3}{\pi}\arccos\left(\frac{4\,k1+3\,k2}{2\,\sqrt{3\,k1+2\,k2}\sqrt{8\,k1z+k2z'}}\right)$$

4.3 Proof of Theorem 3

Proof. To begin, we note that the probability that candidate a is a Condorcet winner and selected for a given scoring rule under IWOC is a quadrivariate normal orthant probability from

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} F(r_1, r_2, r_3, r_4) = P^{\infty}_{CW_a}(\lambda, k_1, k_2),$$

where $F(r_1, r_2, r_3, r_4)$ is the probability density function for the multivariate normal distribution with correlation matrix R. A reduced form of this is obtained by Gehrlein and Valognes [13].

Let $P^{\infty}_{CL_a}(\lambda, k_1, k_2)$ denote the probability that candidate 'a' is both the Condorcet loser and is elected by a weighted rule under IWOC. Then,

$$P^{\infty}_{CL_a}(\lambda, k_1, k_2) = \int_0^{\infty}\int_0^{\infty}\int_{-\infty}^{0}\int_{-\infty}^{0} F(r_1, r_2, r_3, r_4)$$

Let notation of the form $F(r_1, r_2, -, r_4)$ denote the marginal probability that is derived from $F(r_1, r_2, r_3, r_4)$ for r_3, with R modified accordingly (third row and third column removed). It follows that (using a recollection of Sheppard's theorem)

$$\begin{aligned}
P^{\infty}_{CL_a}(\lambda, k_1, k_2) &= \int_0^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{0} F(r_1, r_2, r_3, r_4) \\
&- \int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_{-\infty}^{0} F(r_1, r_2, r_3, r_4) \\
&= \int_0^{\infty}\int_0^{\infty}\int_{-\infty}^{0} F(r_1, r_2, -, r_4) \\
&- \int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} F(r_1, r_2, r_3, r_4) + P^{\infty}_{CW_a}(\lambda, k_1, k_2) \\
P^{\infty}_{CL_a}(\lambda, k_1, k_2) - P^{\infty}_{CW_a}(\lambda, k_1, k_2) &= \int_0^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} F(r_1, r_2, -, r_4) \\
&- \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} F(r_1, r_2, -, r_4) \\
&- \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} F(r_1, r_2, r_3, -) \\
&= \int_0^{\infty}\int_0^{\infty} F(r_1, r_2, -, -) - \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} F(r_1, r_2, -, r_4) \\
&- \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} F(r_1, r_2, r_3, -)
\end{aligned}$$

Using Sheppard's Theorem of Median Dichotomy, this will reduce to a sample function of some "arcsin" terms. Thus, we have a fairly simple form for

$P^{\infty}_{CL_a}(\lambda, k_1, k_2)$. And the difference (loser-winner) is very simple in form. So, it is easy to obtain a representation for $P^{\infty}_{CL_a}(\lambda, k_1, k_2)$:

$$\begin{aligned}\text{Loser} - \text{Winner} &= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} F(r_1, r_2, r_3, -) + \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} F(r_1, r_2, -, r_4) \\ &- \int_0^{\infty}\int_0^{\infty} F(r_1, r_2, r_3, -)\end{aligned}$$

The two- and three-variate orthant probabilities in this representation can be evaluated directly from Sheppard's 1898 Theorem of Median Dichotomy (Kendall and Stuart [15]) to obtain, after reduction:

$$P^{\infty}_{CL_a}(\lambda, k_1, k_2) = P^{\infty}_{CW_a}(\lambda, k_1, k_2) - \left(\frac{\arcsin(\rho) + \arcsin(\rho/2)}{3(\pi/2 + \arcsin(\frac{k1+k2}{3k1+2k2}))}\right)$$

Where $P^{\infty}_{CW_a}(\lambda, k_1, k_2) = \dfrac{\Phi(4, R)}{1/3P^{\infty}_{Con}(IWOC)}$ (Cf Gehrlein and Valognes [13]).

For three candidates, there is a Condorcet winner if and only if, there is a Condorcet loser.

By the symmetry of IWOC we have:

$$P^{\infty}_{CL}(\lambda, k_1, k_2) = 3 \times P^{\infty}_{CL_a}(\lambda, k_1, k_2).$$

□

Computed values of $P^{\infty}_{CL}(\lambda, k_1, k_2)$ are given in Table 3 for some values of λ, $k1$ and $k2$.

5 Concluding Comments

Our results tend to prove that letting voters cast ballots with weak preferences increases the different types of Condorcet efficiency of the scoring rules. To some extent, letting individuals express more varied opinion increases homogeneity! Nevertheless, we don't know to which extent these conclusions remain valid with more candidates. Introducing weak orderings does not change the status of the Borda count: This is still the scoring rule that has more affinities with majority criteria. Further studies could determine whether using other methods to define scoring rules for weak orderings (different from the one Black and Smith suggested) would lead to better results in term of Condorcet efficiency of the scoring rules.

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Table 1: Probabilities $P_{ab}^{\infty}(\lambda, k1, k2)$ and $P_{bc}^{\infty}(\lambda, k1, k2)$

$\mathbf{k1 = 0.1}$	$\lambda = 0$	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.3$	$\lambda = 0.4$	$\lambda = 0.5$
$k2 = 0.0$	0.7553	0.7794	0.8044	0.8282	0.8464	0.8534
$k2 = 0.1$	0.7877	0.8117	0.8365	0.8598	0.8777	0.8846
$k2 = 0.2$	0.8044	0.8284	0.8531	0.8764	0.8945	0.9016
$k2 = 0.9$	0.8386	0.8626	0.8876	0.9123	0.9329	0.9419
$\mathbf{k1 = 0.5}$	$\lambda = 0$	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.3$	$\lambda = 0.4$	$\lambda = 0.5$
$k2 = 0.1$	0.7641	0.7881	0.8131	0.8367	0.8547	0.8617
$k2 = 0.2$	0.7714	0.7955	0.8204	0.8439	0.8618	0.8687
$k2 = 0.5$	0.7877	0.8117	0.8365	0.8598	0.8777	0.8846
$\mathbf{k2 = 0.1}$	$\lambda = 0$	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.3$	$\lambda = 0.4$	$\lambda = 0.5$
$k1 = 0.0$	0.8604	0.8848	0.9111	0.9391	0.9688	1.0000
$k1 = 0.1$	0.7877	0.8117	0.8365	0.8598	0.8777	0.8846
$k1 = 0.2$	0.7747	0.7987	0.8236	0.8470	0.8649	0.8718
$k1 = 0.9$	0.7604	0.7844	0.8095	0.8331	0.8512	0.8582

Table 2: Probabilities $P_{ac}^{\infty}(\lambda, k1, k2)$

$\mathbf{k1 = 0.1}$	$\lambda = 0$	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.3$	$\lambda = 0.4$	$\lambda = 0.5$
$k2 = 0.0$	0.9016	0.9223	0.9409	0.9557	0.9654	0.9688
$k2 = 0.1$	0.9288	0.9457	0.9603	0.9717	0.9789	0.9814
$k2 = 0.2$	0.9409	0.9558	0.9686	0.9784	0.9846	0.9868
$k2 = 0.9$	0.9614	0.9729	0.9824	0.9896	0.9941	0.9956
$\mathbf{k1 = 0.5}$	$\lambda = 0$	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.3$	$\lambda = 0.4$	$\lambda = 0.5$
$k2 = 0.1$	0.9095	0.9292	0.9466	0.9604	0.9694	0.9725
$k2 = 0.2$	0.9158	0.9346	0.9511	0.9641	0.9725	0.9754
$k2 = 0.5$	0.9288	0.9457	0.9603	0.9717	0.9789	0.9814
$\mathbf{k2 = 0.1}$	$\lambda = 0$	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.3$	$\lambda = 0.4$	$\lambda = 0.5$
$k1 = 0.0$	0.9719	0.9815	0.9893	0.9952	0.9988	1.0000
$k1 = 0.1$	0.9288	0.9457	0.9603	0.9717	0.9789	0.9814
$k1 = 0.2$	0.9185	0.9369	0.9530	0.9657	0.9739	0.9767
$k1 = 0.9$	0.9062	0.9263	0.9442	0.9585	0.9677	0.9710

Table 3: Probabilities $P_{CL}^{\infty}(\lambda, k1, k2)$

$\mathbf{k1 = 0.1}$	$\lambda = 0$	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.3$	$\lambda = 0.4$	$\lambda = 0.5$
$k2 = 0.0$	0.03709	0.02383	0.01261	0.00462	0.00068	0
$k2 = 0.1$	0.02883	0.01831	0.00962	0.00352	0.00052	0
$k2 = 0.2$	0.02535	0.01607	0.00847	0.00312	0.00047	0
$k2 = 0.9$	0.01971	0.01265	0.00686	0.00269	0.00044	0

$\mathbf{k1 = 0.5}$	$\lambda = 0$	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.3$	$\lambda = 0.4$	$\lambda = 0.5$
$k2 = 0.1$	0.03465	0.02217	0.01170	0.00427	0.00063	0
$k2 = 0.2$	0.03273	0.02088	0.01099	0.00402	0.00059	0
$k2 = 0.5$	0.02883	0.01831	0.00962	0.00352	0.00052	0

$\mathbf{k2 = 0.1}$	$\lambda = 0$	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.3$	$\lambda = 0.4$	$\lambda = 0.5$
$k1 = 0.0$	0.01715	0.01130	0.00651	0.00294	0.00074	0
$k1 = 0.1$	0.02883	0.01831	0.00962	0.00352	0.00052	0
$k1 = 0.2$	0.03191	0.02034	0.01070	0.00391	0.00058	0
$k1 = 0.9$	0.03566	0.02286	0.01207	0.00441	0.00065	0