The Euler Line in Hyperbolic Geometry

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Abstract- In Euclidean geometry, the most commonly known system of geometry, a very interesting property has been proven to be common among all triangles. For every triangle, there exists a line that contains three major points of concurrence for that triangle: the centroid, orthocenter, and circumcenter. The centroid is the concurrence point for the three medians of the triangle. The orthocenter is the concurrence point of the altitudes. The circumcenter is the point of concurrence of the perpendicular bisectors of each side of the triangle. This line that contains these three points is called the Euler line. This paper discusses whether or not the Euler line exists in Hyperbolic geometry, an alternate system of geometry. The Poincare Disc Model for Hyperbolic geometry, together with Geometer's Sketchpad software package was used extensively to answer this question.

Introduction

In Euclidean geometry, the most commonly known system of geometry, a very interesting property has been proven to be common among all triangles. For every triangle, there exists a line that contains three major points of concurrence for that triangle: the centroid, orthocenter, and circumcenter. The centroid is the concurrence point for the three medians of the triangle. The orthocenter is the concurrence point of the altitudes. The circumcenter is the point of concurrence of the perpendicular bisectors of each side of the triangle. This line that contains these three points is called the Euler line (For the Euclidean result see Wallace, 1998, p. 174). This paper discusses whether or not the Euler line exists in Hyperbolic geometry, an alternate system of geometry. The Poincare Disc Model for Hyperbolic geometry, together with Geometer's Sketchpad software package was used extensively to answer this question. The Poincare Disc Model allows only points that lie in the interior of the unit circle. Lines appear in two forms. First, all diameters of the unit circle, excluding endpoints, are lines. Second, for any circle that is orthogonal to the unit circle, the points on that circle lying in the interior of the unit circle will be considered a hyperbolic line (Wallace, 1998, p. 336).

Methods

Locating the Centroid

In attempting to locate the Euler line, the first thing that needs to be found is the midpoint of each side of the hyperbolic triangle. This will lead to the construction of the medians, and therefore the centroid. In Figure 1, the hyperbolic triangle ABC is within the unit circle (Poincare's disc). Note that sides AB and AC are segments of diameters, and side BC is an arc of the necessary circle of inversion (a circle orthogonal to the unit circle).

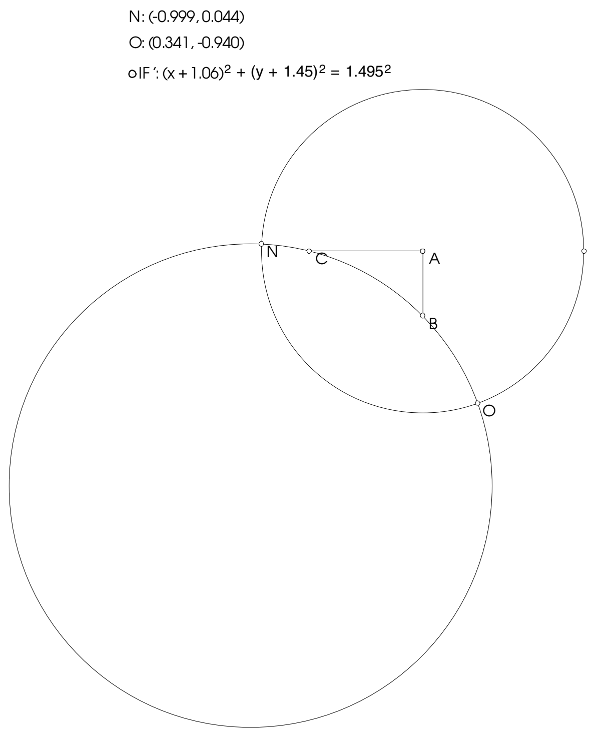

Figure 1.

a) We will make use of Poincare's distance formula,

$$d(A,B) = \left|\ln\left(\frac{AQ}{BQ}\right)\times\left(\frac{BP}{AP}\right)\right|,$$

where Q and P are points of intersection of the hyperbolic line with the unit circle.

For a side of a triangle which is the segment of a diameter of the unit circle let the vertices be A(0,0), B(0,-0.4) and

$$d(A,B) = \left|\ln\left(\frac{7}{3}\right)\right|$$

Let M be the midpoint of side AB.

$$d(A,M) = \left|\ln\left(\frac{1+M}{1-M}\right)\right| = \frac{1}{2}\ln\frac{7}{3},$$

therefore:

$$\frac{1+M}{1-M} = 1.527525\text{, and}$$

M=-0.208712.

Since the segment AB runs along the y-axis, M is the y coordinate of AB.

b) For a side of a triangle which is the arc of a circle orthogonal to the unit circle, we let the vertices be B(0,-0.4), C(-0.7,0), and *d*(*B*, *C*) = 2.07.

To locate the coordinates of the midpoint M_2, (x, y), of side BC, we need to solve a system of two equations with two unknowns.

If we use the Poincare distance formula, and the points N and O which are the intersections of the hyperbolic line containing BC and the unit circle, we find an equation in x and y.

$$d(B,Midpt) = \left|\ln\left(\frac{MO}{BO}\times\frac{BN}{MN}\right)\right| = 1.035$$

$$= \left|\ln\left(\frac{\sqrt{(x-.341)^2+(y-.940)^2}}{1.416}\times\frac{2.423}{\sqrt{(x+.999)^2+(y-.044)^2}}\right)\right| = 1.035$$

And using the equation of the circle containing hyperbolic line BC:

$$(x+1.06)^2 + (y+1.45)^2 = 1.495^2$$

we have a second equation in x and y.

Solving these two equations yields:

$x=-0.3659746648$

$y=-0.125857321$

These are the coordinates of the midpoint M_2 of side BC.

Once the midpoint of each side of the triangle has been found, the centroid can be found quite easily. Construct each of the medians of the triangle, and the common point of intersection of the three medians is the centroid. The centroid will occur in all hyperbolic triangles. The median concurrence theorem has been proven in neutral geometry, or independent of any axioms of parallelism.

Locating the Orthocenter

The next step is to locate the orthocenter of the triangle. The orthocenter is the concurrence point of the three altitudes of a triangle. This can also be found using constructions on the Geometer's Sketchpad program (see Figure 2).

Find the altitude from vertex E to side CD:

a) Find E", the image of E under a transformation of inversion on the circle that contains segment CD. The transformation of inversion is a method that helps in locating the hyperbolic lines that are arcs of circles orthogonal to the unit circle. Any circle containing both the image A' of a point A under inversion and A is orthogonal to the circle of inversion (Wallace, 1998, p.284). Thus if you wish to find the hyperbolic line that contains both A and B, find A', and the unique circle containing A, B, and A' would be orthogonal to the unit circle.

b) Find the hyperbolic line that connects E to E".

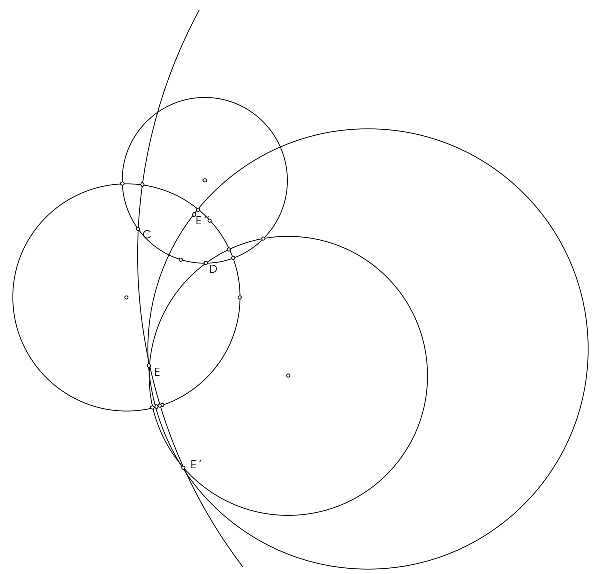

Figure 2.

By definition, this will construct a line through E that is perpendicular to CD. This line will also contain E', the image of E under a transformation of inversion on the unit circle, since the line must be perpendicular to the unit circle. This line is therefore the altitude. The altitude from each vertex can be found in a similar manner. The common point of intersection for the three altitudes of a triangle is the orthocenter. While no apparent counterexample of the existence of the orthocenter for hyperbolic triangles has been found, the orthocenter has not been proven to exist for all triangles.

Locating the Circumcenter

Next, we need to locate the circumcenter. The circumcenter is the concurrence point of the three perpendicular bisectors of a triangle. This can be found using constructions on the Geometer's Sketchpad program (see Figure 3). The following construction will create the hyperbolic line through midpoint M that is orthogonal to the segment that contains M.

a) Construct the radius OM of the circle that contains the hyperbolic line on which the midpoint M lies.
b) Construct the line perpendicular to the circle at M.
c) Do the same for M', the image of M under the transformation of inversion on the unit circle.
d) The intersection of these two tangent lines will be the center of the circle that contains the perpendicular bisector of side CD.

This is true, since the circle found through this construction bisects CD since it goes through point M, and is perpendicular to CD since it is perpendicular to the circle containing CD. This circle is also perpendicular to the unit circle, since it contains both M and M', complying with the rules of the Poincare disc model.

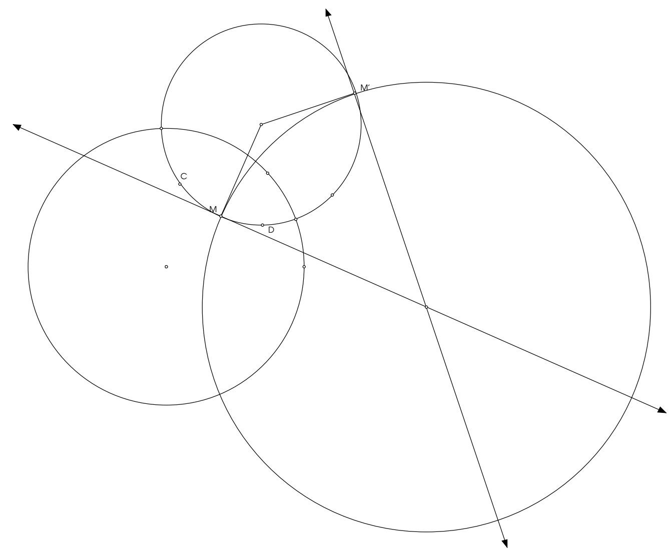

Figure 3.

Locating the perpendicular bisector for each side of a triangle in this manner, we find the common point of intersection of the perpendicular bisectors called the circumcenter. Unfortunately, the circumcenter does not exist for all hyperbolic triangles. A counterexample can be seen in Figure 4. The hyperbolic triangle has vertices C(0, 0), D(0.9, 0), and E(0, 0.8). The three perpendicular bisectors for the sides of the triangle do not intersect at all; thus the circumcenter does not exist.

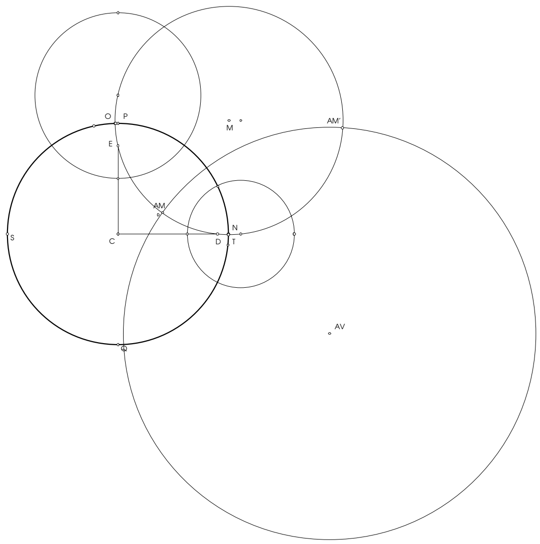

Figure 4

Results

Once a hyperbolic triangle with a centroid, circumcenter, and orthocenter is found, the colinearity of these points needs to be examined to determine if the Euler line exists (See figure 5). The hyperbolic triangle has the vertices A(0, 0), B(0, -0.4), and C(-0.7, 0). The centroid and circumcenter have been constructed and are labeled Cen and Circ respectively. The orthocenter for this triangle is vertex A. Since A is the origin, if the Euler line exists it must be a diameter of the circle. It is obvious that A, Cen, and Circ are not collinear. Thus we have shown a counterexample to the Euler line in hyperbolic geometry.

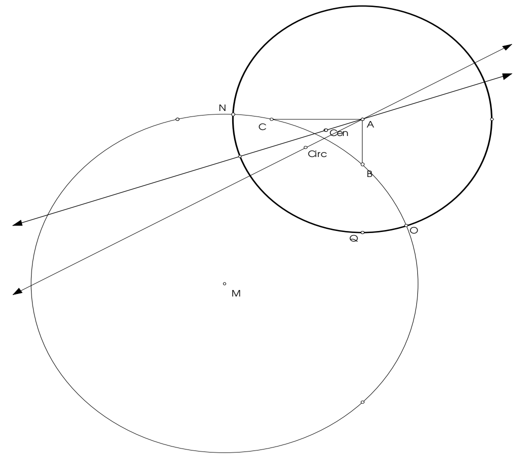

Figure 5

We have also seen another counterexample, in the case where the circumcenter did not exist (Figure 4). In this research, no hyperbolic triangle has been found to contain an Euler line. In other words, in hyperbolic geometry, the centroid, circumcenter, and orthocenter are not collinear. There is one exception to this statement. In an equilateral triangle, the centroid, circumcenter, and orthocenter are the same point. They are therefore collinear, and the Euler line exists. The fact that every equilateral triangle contains an Euler line can be proven neutrally, and is therefore true for both Euclidean and Non-Euclidean geometry. In summary, this research found no evidence of the existence of the Euler line in hyperbolic geometry. In fact, a counterexample was found for all types of triangles, with the exception of the equilateral triangle.

References:

Wallace, Edward C., and Stephen F. West, *Roads to Geometry, 2nd ed*. Upper Saddle River, NJ: Prentice Hall, 1998.

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